THE OCTOBER MEETING IN NEW YORK

The two hundred thirty-seventh regular meeting of the Society was held at Columbia University on Saturday, October 25, 1924, extending through the usual morning and afternoon sessions. The attendance included the following fifty members of the Society:

Alexander, Archibald, Babb, C. R. Ballantine, J. P. Ballantine, Barnum, W. M. Bond, R. W. Burgess, B. H. Camp, Cole, L. D. Cummings, Dantzig, Doak, Eisenhart, Fite, Fort, Philip Franklin, Gafafer, Garretson, Gehman, Gill, Glenn, Gronwall, Hille, Himwich, Dunham Jackson, Joffe, Kline, Langman, Lefschetz, Longley, MacColl, Meder, C. L. E. Moore, Northcott, Pell, R. G. Putnam, Rainich, Reddick, R. G. D. Richardson, Ritt, Ruger, Seely, C. E. Smith, J. H. Taylor, J. M. Thomas, Veblen, M. E. Wells, Wiener, R. G. Wood.

At the May Meeting of the Society a new type of membership was created and designated Sustaining Membership. This action was taken on the conviction that there are individuals and institutions who realize the fundamental part that mathematical research plays in our modern civilization and who are willing to render financial assistance to its publication. At the present meeting the Secretary took great pleasure in announcing the following list of Sustaining Members who had been solicited by the Committee on Endowment:

Allyn and Bacon, Boston;
Babcock and Wilcox Company, New York;
Dartmouth College, Hanover;
General Electric Company, Schenectady; Patron;
Ginn and Company, Boston;
Insull Interests, Chicago (comprising the Commonwealth Edison Company, the People's Gas Light and Coke Company, the Public Service Company, the Middle West Utilities Company, and the Chicago, North Shore and Milwaukee Railroad); Patron;
Mr. E. W. Rice, Jr., General Electric Company, Schenectady;
Union Central Life Insurance Company, Cincinnati;
University of Washington, Seattle;
Western Electric Company, New York; Patron;
Westinghouse Electric and Manufacturing Company, New York; Patron.

The Secretary announced also that the following persons had been elected to ordinary membership in the Society by

the Council since April, 1924, either by mail vote* or at a meeting:

Mr. Anantanarayana Aiyar, Judson College, Rangoon;
Mr. William Leake Ayres, University of Texas;
Miss Wealthy Babcock, University of Kansas;
Mr. Frank Swan Beale, University of Maine;
Professor Ella Edna Bernstorf, Friends University;
Professor Harold Blair, Western State Normal College, Kalamazoo;
Mr. Leonard Marcot Blumenthal, University of Michigan;
Mr. Robert Bogue, Omaha;
Professor Zechariah Chafee, Harvard University Law School;
Professor Tsaihsin H. Chen, Yen-Ching University, Peking;
Mr. Starling W. Childs, Jr., Yale University;
Professor Jung Lu Chin, Nankai University, Tientsin;
Superintendent William Horner Cocke, Virginia Military Institute;
Mr. Redmond S. Colnon, St. Louis;
Miss Mae Elizabeth Conn, University of Southern California;
Sister Mary Cordia, Notre Dame College;
Professor Moses Eugene Cox, Agricultural and Mechanical College of Texas;
Professor Marian Elizabeth Daniells, Iowa State College;
Professor Robert Dodds Daugherty, Iowa State Teachers College;
Professor Raymond Mark Deming, Upper Iowa University;
Mr. Howard Theodore Engstrom, University of Maine;
Professor Claire Beatrice Fisher, Potomac State College;
Mr. Donald Alexander Flanders, University of Pennsylvania;
Mr. Roger Lee Flanders, Norwich University;
Mr. Alfred Douglas Flinn, Engineering Foundation, New York;
Miss Neda Belle Freeman, Irving College;
Professor Benjamin Curtis Glover, Otterbein College;
Professor George Wollam Gorrell, University of Denver;
Professor Vernon Guy Grove, Michigan Agricultural College;
Mr. Ira James Gwinn, Morningside College;
Professor John Alexander Hardin, Centenary College of Louisiana;
Mr. Floyd S. Harper, University of Nebraska;
Professor Nathan W. Harter, Thiel College;
Professor Frederick Charles Hartwick, University of Dayton;
Mr. Hubert Banks Huntley, University of New Hampshire;
Miss Mabel Hutchins, Blue Mountain College;
Mr. Saburo Isayama, Niigata High School for Girls;
Professor Ralph Lent Jeffery, Acadia University;
Professor Elijah Newton Johnson, Butler College;
Professor Jesse Brelande Johnson, Baylor University;
Miss Dora Evelyn Kearney, University of Minnesota;
Mr. Ernest George Keller, University of Wisconsin;
Mr. Arthur J. Koeppe, Union Central Life Insurance Company, Cincinnati;
Mr. Abraham Boris Kouperman, University of Chicago;
Mr. William H. Kurzin, Crane Junior College;
Mr. Benjamin George Lehenbauer, Union Central Life Insurance Company, Cincinnati;
Mr. John J. Lichter, St. Louis;
Mr. Hugh Gray Lieber, Columbia University;

* See this BULLETIN, vol. 30, p. 481.

Professor William Thomas MacCreadie, Norwich University;
Professor Anna E. Many, Newcomb College;
Mr. Thomas Midgley, General Motors Chemical Company, Dayton;
Professor Kevin Nowlan, S. J., Loyola University;
Professor Yoshitomo Okada, Tôhoku Imperial University;
Professor Earle Kerr Paxton, Washington and Lee University;
Mr. Walter Otis Pennell, Southwestern Bell Telephone Company, St. Louis;
Professor Sue Scott Perkins, Missouri State Teachers College, Springfield;
Mr. Howard Manchester Phillips, DeKalb Junction, N. Y.;
Dr. Robert Garfield Putnam, New York University;
Miss Margaret Ramsey, Linfield College;
Professor Caroline M. Reaves, Coker College;
Dr. Bernard Paul Reinsch, University of Illinois;
Dr. Norman Hurd Ricker, Humble Oil and Refining Company, Houston;
Professor Joseph Alphonsus Rooney, Mount St. Charles College;
Mr. Norman Eby Rutt, University of Pennsylvania;
Professor Samuel Thomas Sanders, Louisiana State University;
Dr. Paul Schnurmann, Pittsburgh;
President John Stephen Sewell, Alabama Marble Company, Birmingham;
Professor Charles E. Shull, Bridgewater College;
Professor Elmer McClellan Stahl, Midland College;
Professor Will B. Stokes, Southern Louisiana Institute;
Miss Lucretia Mae Switser, University of Kansas;
Dr. Brandreth Symonds, Mutual Life Insurance Company;
Mr. Keizo Takahashi, Shonan Middle School, Banagawa-Ken;
Professor William Duane Tallman, Montana State College;
Mr. Ward Hastings Taylor, University of Arkansas;
Mr. Benjamin Franklin Tillson, New Jersey Zinc Company;
Mr. Paul S. Wagner, Johns Hopkins University;
Miss Esther May Weaver, Northwestern University;
Dr. David Vernon Widder, Bryn Mawr College;
Mr. Frank Harry Mead Williams, Drexel Institute.

Ordinary Members nominated by Sustaining Members:

Mr. W. L. Abbott, Commonwealth Edison Company, Chicago;
Mr. R. J. Andrus, Twin State Gas and Electric Company, Boston;
Mr. C. C. Argabrite, Interstate Public Service Company, Indianapolis;
Mr. H. E. Bates, Peoples Gas Light and Coke Company, Chicago;
Mr. L. V. Bewley, General Electric Company, Schenectady;
Mr. J. E. Black, Babcock and Wilcox Company, New York;
Mr. E. J. Blair, Chicago Rapid Transit Company;
Mr. L. F. Blume, General Electric Company, Pittsfield;
Mr. G. T. Bogard, Kentucky Utilities Company, Louisville;
Mr. Aram Boyajian, General Electric Company, Pittsfield;
Mr. B. O. Buckland, General Electric Company, Schenectady;
Mr. B. I. Budd, Chicago, North Shore and Milwaukee Railroad Company;
Mr. F. H. Buller, General Electric Company, Pittsfield;
Mr. C. M. Burrill, General Electric Company, Schenectady;
Mr. J. R. L. Carrington, Union Central Life Insurance Company, Cincinnati;
Mr. P. R. Cassidy, Babcock and Wilcox Company, New York;
Miss E. Clarke, General Electric Company, Schenectady;
Mr. S. B. Cushing, Public Service Company of North Illinois, Chicago;
Mr. G. F. Davis, General Electric Company, Schenectady;

Dr. Saul Dushman, General Electric Company, Schenectady;
Mr. J. W Evers, Commonwealth Edison Company, Chicago;
Mr. C. L. Fortescue, Westinghouse Company, East Pittsburgh;
Mr. E. J. Fowler, Commonwealth Edison Company, Chicago;
Mr. R. F. Franklin, General Electric Company, Schenectady;
Mr. G. W. Hamilton, Middle West Utilities Company, Chicago;
Mr. E. E. Hardcastle, Union Central Life Insurance Company, Cincinnati;
Mr. R. B. Harper, Peoples Gas Light and Coke Company, Chicago;
Mr. Isaac Harter, Babcock and Wilcox Company, New York;
Mr. J. L. Hecht, Public Service Company of North Illinois, Chicago;
Mr. G. J. Heimberger, Babcock and Wilcox Company, Bayonne;
Mr. Walter Helmer, Chicago Rapid Transit Company;
Mr. R. E. Hellmund, Westinghouse Company, East Pittsburgh;
Mr. Jesse Hyatt, Chicago, North Shore and Milwaukee Railroad Company;
Mr. E. E. Johnson, General Electric Company, Schenectady;
Mr. H. A. Johnson, Chicago Rapid Transit Company;
Mr. Charles Jones, Chicago Rapid Transit Company;
Mr. W. A. Jones, Babcock and Wilcox Company, Bayonne;
Mr. E. W. Kellogg, General Electric Company, Schenectady;
Mr. H. J. Kerr, Babcock and Wilcox Company, New York;
Mr. A. L. Kimball, General Electric Company, Schenectady;
Mr. C. H. Koch, General Electric Company, Schenectady;
Mr. A. J. Koeppe, Union Central Life Insurance Company, Cincinnati;
Mr. L. L. Langvand, Babcock and Wilcox Company, Barberton;
Mr. E. S. Lee, General Electric Company, Schenectady;
Mr. N. E. Lewis, Babcock and Wilcox Company, New York;
Mr. D. W. McLenegan, General Electric Company, Schenectady;
Mr. Campbell MacMillan, General Electric Company, Schenectady;
Mr. H. B. Marvin, General Electric Company, Schenectady;
Mr. S. T. Maunder, General Electric Company, Schenectady;
Mr. M. S. Mead, General Electric Company, Schenectady;
Mr. F. W. Merrill, General Electric Company, Fort Wayne;
Mr. C. W. Middletown, Babcock and Wilcox Company, New York;
Mr. Harold Mott-Smith, General Electric Company, Schenectady;
Mr. Harold Otis, Chicago Rapid Transit Company;
Mr. W. E. Paul, General Electric Company, Schenectady;
Mr. C. W. PenDell, Public Service Company of North Illinois, Chicago;
Mr. J. F. Peters, Westinghouse Company, East Pittsburgh;
Mr. A. D. Pratt, Babcock and Wilcox Company, New York;
President A. G. Pratt, Babcock and Wilcox Company;
Mr. C. W. Rice, Schenectady;
Mr. J. D. Roberts, Central Illinois Public Service Company, Springfield;
Mr. E. L. Robinson, General Electric Company, Schenectady;
Mr. R. F. Schuchardt, Commonwealth Edison Company, Chicago;
Mr. E. O. Schweitzer, Commonwealth Edison Company, Chicago;
Mr. E. B. Shand, Westinghouse Company, East Pittsburgh;
Mr. I. H. Summers, General Electic Company, West Lynn;
Mr. H. D. Taylor, General Electric Company, Schenectady;
Mr. C. E. Thompson, Chicago, North Shore and Milwaukee Railroad Company;
Mr. Stephan Timoshenko, Westinghouse Company, East Pittsburgh;
Mr. L. A. Umansky, General Electric Company, Schenectady;
Mr. G. B. Warren, General Electric Company, Schenectady;
Mr. Hosea Webster, Babcock and Wilcox Company, New York;
Mr. R. W. Wieseman, General Electric Company, Schenectady.

During the summer, the Council voted thanks for the following gifts to the Endowment Fund: from Stone and Webster, $1000; from the Cutler-Hammer Company, $250; from Mr. C. H. Coffin, ex-president and founder of the General Electric Company, $2000. It was voted that sustaining members may be elected by the Council at the time of the first presentation of their names, and that not more than five persons may be nominated to ordinary membership by a sustaining member for each $100 of dues paid.

The President made the following appointments to represent the Society: at the Pan-American Scientific Congress at Lima, Professors E. V. Huntington and D. N. Lehmer; at the inauguration of President Vinson of Western Reserve University, Professor W. G. Simon; at the celebration of the one hundredth anniversary of the Rensselaer Polytechnic Institute, Dr. G. A. Campbell; at the International Mathematical Congress at Toronto, Professor G. A. Bliss and the Secretary; at the inauguration of President Irving Maurer, of Beloit College, Professor H. H. Conwell.

The President appointed as Committee to Administer the Eliakim Hastings Moore Fund, Professors Arnold Dresden (chairman), J. W. Alexander, and H. E. Slaught. Mr. Robert Henderson, of the Equitable Life Assurance Society, was designated to deliver the second Josiah Willard Gibbs Lecture at the coming Annual Meeting in Washington; his subject will be *Life insurance as a social service and as a mathematical problem.*

A committee consisting of Mr. S. A. Joffe and Professor Tomlinson Fort was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers, members of the Council and members of the Board of Trustees was adopted and ordered printed on the official ballot for the annual election. An invitation was received from the University of Pennsylvania to hold the summer meeting at that institution in 1926.

The report of the Committee on the Cole Prize (see this BULLETIN, vol. 29, p. 14) recommending that the prize be offered for the most notable advance in the theory of Linear Algebras was adopted. Further details will be published shortly by the Committee of which Professor H. S. White is chairman.

It has been decided to continue the printing of the BULLETIN and TRANSACTIONS in Hamburg for the year 1925.

Titles and abstracts of the papers read at this meeting follow below. President Veblen presided at both sessions. Mr. Huber was introduced by Professor G. E. Wahlin. The papers of Alexander, Cole, Franklin (second paper), Garabedian, Hille (first paper), Huber, Hutchinson, Michal, Pennell, Walsh, and Williams were read by title.

1. Professor B. H. Camp: *Note on a transformation of the hypergeometric series.*

The sum of the first t terms of the hypergeometric series which occurs in the theory of probability is* $(1/n^{(r)}) \sum_{t=0}^{t=t} (r^{(t)}/t!)\, y^{(s)}\, (n-y)^{(t)}$, where $s = r-t$, and $r^{(t)} = r(r-1)\cdots(r-t+1)$. This may be transformed to the form: $(r^{(t+1)}/n^{(r)}t!)\sum_{y=s-1}^{y=y-1} y^{(s-1)}\,(n-y-1)^{(t)}$, the summation now being with respect to y. The process by which this is obtained may be of value in the transformation of other sums. The details will appear in a paper on *Probability integrals for a hypergeometric series*, in BIOMETRIKA.

2. Professor Louise D. Cummings: *A new type of double sextette closed under a binary* (3,3) *correspondence.*

This paper shows that cyclic double sextettes closed under a binary (3,3) correspondence are reducible to two non-congruent types. Type I was determined in a previous paper, and type II is here exhibited.

3. Dr. J. H. Taylor:† *Reduction of Euler's equations to a canonical form.*

This paper will appear in full in an early issue of this BULLETIN.

* Cf. Elderton, *Frequency Curves and Correlation*, p. 37.

† National Research Fellow.

4. Mr. H. M. Gehman: *Necessary and sufficient conditions that every closed and connected subset of a continuous curve be a continuous curve. Second paper.*

The conditions previously announced to the Society (Chicago Meeting, April 19, 1924) are contained in this more general theorem: *A necessary and sufficient condition that every closed and connected subset of a continuous curve M be a continuous curve is that M contain no closed and connected point set N, containing a continu of condensation W, such that $N-W$ is uniformly connected im kleinen "relative to W."* Another necessary and sufficient condition is that, given any positive number ε, M contains at most a finite number of mutually exclusive closed and connected sets of diameter greater than ε. A condition which is sufficient but not necessary is that, given any positive number ε, M can be expressed as the sum of a finite number of closed and connected sets each of diameter less than ε, each pair having at most a finite number of points in common.

5. Dr. Philip Franklin: *The rotating disc.*

A homogeneous incompressible disc of radius r, when caused to rotate with constant angular velocity ω, suffers a contraction in radius, according to the theory of relativity. Eddington, assuming that the geometry on the disc remains euclidean, finds the new radius to be, approximately, $r' = r(1-\omega^2 r^2/8)$. The author discusses the results when this assumption is dropped, certain others being retained, and finds for the new radius approximately $r' = r(1-\omega^2 r^2/14)$. Some properties of the geometry on the disc are also given.

6. Dr. Philip Franklin: *The electric currents in a network.*

If constant E. M. F.'s are inserted in a network of conductors, the resulting currents are uniquely determined. Their explicit expression in terms of the resistances of the conductors and the topological properties of the network was given by Kirchhoff in 1847. In this paper the result is obtained by a briefer method, based on the matrices for a linear graph.

7. Professor F. N. Cole: *On simple groups of low order.*

This paper appeared in full in the November-December, 1924, number of this BULLETIN.

8. Professor J. I. Hutchinson: *On the roots of the Riemann zeta function.*

The object of this paper is to simplify the methods and formulas in use in connection with numerical investigations on the zeta function. The limits of an empirical law noticed by Gram, regarding the separation of the roots, are determined and the results used in locating 59 additional roots, thus completing the list of those smaller than 300. The values of 14 roots are calculated, thus including the remaining unknown roots smaller than 100. The number of positive roots less than 500 is calculated and found to be 269.

9. Professor Einar Hille: *A note on regular singular points.*

This paper contains a discussion of the differential equation $w'' + G(z)w = 0$ in the neighborhood of the origin, which is supposed to be a regular singular point. The discussion is based upon the transformation $Z(z; z_0) = \int_{z_0}^{z} G(z)dz$; $W(Z) = [G(z)]^{-1/4}w(z)$, and involves among other things a characterization of the nature of the singular point by means of the geometrical singularity at the origin of the curve net $X = \text{const.}$, $Y = \text{const.}$ The orientation of the zeros and the extrema of the solutions upon a Riemann manifold in the neighborhood of the origin is studied, and regions are determined in which these points are regularly distributed. The results are incomplete in the case in which one of the roots of the indicial equation is greater than $+1$. The investigation is completed with a discussion based on the method of zero-free regions.

10. Professor C. A. Garabedian: *Four methods for solving the problem of the rectangular beam.*

Upon applying to beams of rectangular section the series already employed by the author in connection with plates and rods*, one is led to four distinct methods. Method A deals with the broad shallow beam, while method B is concerned with the beam that is narrow and deep. Methods AA and BB (announced in the closing paragraph of the author's note on *Thick rectangular plates*†) are

* See abstract in this BULLETIN, vol. 30, p. 295, with accompanying references.

† COMPTES RENDUS, Feb. 11, 1924, and this BULLETIN, loc. cit.

analogous to methods A and B respectively. All four methods treat the beam rigorously and in three dimensions, despite the two-dimensional aspect of the problems. Methods A and B excel in rigor, yet AA and BB yield interesting solutions expressible in polynomial form. It is possible to vary the depth in A and the width in B, and in methods AA and BB both width and depth may be varied. A note concerning these methods appeared in the COMPTES RENDUS of August 18, 1924; the author hopes later to publish the further developments in extenso.

11. Professor J. L. Walsh: *Some two-dimensional loci.*

This paper solves the following problem: If two points of the plane, A and B, have as their respective loci the interiors of two circles, determine the locus of the point P such that (I) the angle APB is a prescribed constant, (II) the ratio $PA : PB$ is a prescribed constant. In each case, the locus of P is a region bounded in whole or in part by a cartesian oval or an arc thereof.

12. Professor K. P. Williams: *A uniqueness theorem for the Legendre and the Hermite polynomials.*

Let

$$\varphi(y) = a_0 + a_1 y + (a_2/2!)y^2 + (a_3/3!)y^3 + \cdots,$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x)z + P_2(x)z^2 + \cdots.$$

In this paper it is shown that a simple recursion relation between three successive polynomials of the set $P_0, P_1, P_2, \cdots$ will exist only in case we have to do essentially with the generalized polynomials of Legendre or the polynomials of Hermite.

13. Mr. C. M. Huber: *On the prime divisors of the cyclotomic functions.*

In this paper the author deals with a generalization of the test of Sylvester as to the character of the primes as divisors or non-divisors of the cyclotomic functions. As preliminary, a theorem is given for determining the prime ideal factorization of the rational primes in cyclotomic sub-fields. A necessary and sufficient condition is then developed for a rational prime to be a divisor or non-divisor of the cyclotomic functions. Particular examples are then discussed in application of the test found.

14. Mr. A. D. Michal: *Functional invariants, with a continuity of order p, of one-parameter Fredholm and Volterra transformation groups.*

This paper considers functionals of a function $y(x)$ and its first p derivatives $y'(x), \cdots, y^{(p)}(x)$ admitting a given arbitrary one-parameter *Fredholm* group of linear functional transformations. Sufficient conditions for invariance are given in the form of completely integrable functional equations with partial functional derivatives when the kernel of the transformation is of a certain type. In the case of the linear functional necessary and sufficient conditions for invariance are given. The paper concludes with a theorem showing the unique rôle played by a linear functional of $y(x)$ and its derivative $y'(x)$ in the invariant theory of linear functionals, of continuity order p, which admit a given arbitrary one-parameter *Volterra* group of linear functional transformations.

15. Professor J. W. Alexander: *On certain new topological invariants.*

The author derives certain topological invariants analogous to but in general different from the Betti numbers and coefficients of torsion.

16. Mr. W. O. Pennell: *The interpretation of non-integral exponents with notes on the theory of subponents.*

In this paper the author makes use of the following vector notation: a_α represents a vector in the XY plane of length a making an angle α with the X axis. He calls α the subponent of the vector X, and points out that in multiplication subponents follow laws similar to those of exponents.

17. Professor Norbert Wiener: *The representation of bounded functions by trigonometric integrals.*

The author investigates the condition under which a bounded function may have a trigonometric expansion analogous to a Fourier integral. He studies the convergence in the mean of this expansion both over a finite range and (in the sense of Bohr) over an infinite range.

18. Professor Einar Hille: *A general type of singular points.*

The present paper gives a unified treatment of a fairly general type of singular points of linear differential equations of the second order, based upon the transformation of Liouville. It is assumed that there exists a region abutting upon the singular point which is carried over into a right half-plane by the transformation and, further, that in this half-plane the transformed equation is asymptotic (in an easily characterized sense) to a sine-equation. Such an equation can be integrated by the method of successive approximations and the form of the integral gives directly the asymptotic properties of solutions of the transformed equation in the half-plane. In particular, the distribution of the zeros can be read off easily. Any irregular singular point can be handled by this method; also a wide class of regular singular points, though not all. Further, any singular point which is transformable into one of the previously mentioned types by a change of independent variable can be attacked directly by this method, the detour over the intermediate transformation being unnecessary.

19. Professor Dunham Jackson: *The geometry of frequency functions.*

This paper has appeared in the January-February issue of this BULLETIN.

20. Professor J. F. Ritt: *Elementary functions and their inverses.*

The chief item of this paper is the determination of all elementary functions whose inverses are elementary. The elementary functions are understood to be those obtained by performing algebraic operations and taking exponentials and logarithms. For instance, the function $\tan\left[e^{z^2}-\log_z(1+\sqrt{z})\right]+[z^z+\log\arcsin z]^{1/2}$ is elementary. It is proved that *if $F(z)$ and its inverse are both elementary, there exist n functions $\varphi_1(z)$, $\varphi_2(z)$, $\cdots$, $\varphi_n(z)$, where each $\varphi(z)$ with an odd index is algebraic, and each $\varphi(z)$ with an even index is either e^z or $\log z$, such that $F(z)=\varphi_n\varphi_{n-1}\cdots\varphi_2\varphi_1(z)$, each $\varphi_i(z)$ $(i<n)$ being substituted for z in $\varphi_{i+1}(z)$.*

This paper is a continuation of Liouville's work of almost a century ago on the classification of the elementary functions, on the possibility of effecting integrations in finite terms, and on the impossibility of solving certain differential equations, and certain transcendental equations, in finite terms.

21. Professor J. F. Ritt: *On the impossibility of solving certain differential equations in finite terms.*

Liouville, in 1841, proved that the solutions of Bessel's equation $z^2w'' + zw' + (z^2 - \nu^2)w = 0$ cannot be expressed in terms of z by performing elementary operations* and quadratures. But it is customary to regard a differential equation as solved if one finds a finite equation in w and z; one does not generally try to express w itself in finite terms. In the present paper it is shown that the solution of Bessel's equation does not satisfy any equation in w and z found by performing elementary operations upon those variables and integrating exact differentials.

22. Mr. G. Y. Rainich: *On singularities in analytical physics.*

The introduction, in place of the electromagnetic tensor and its dual, of their sum w, which is a complex tensor, and of complex vectors simplifies calculations. The energy tensor is the norm of w, i. e., w multiplied by its conjugate. Residues of an electromagnetic field are introduced as values of surface integrals surrounding singular lines (electron paths); they are complex numbers only the moduli of which are determined by space-time. Residues can be used to complete the determination of the electromagnetic field, which, in a region free from singularities, is given, by the curvature of space-time, only to a constant of integration (see the author's paper read before the Society in May, 1924). To determine this constant we can agree to make all the residues real, which can be done because the arguments of the residues of every electromagnetic field differ only by multiples of π, proved by the fact that magnetic charges do not exist. The analogy of the electromagnetic field with the analytic function on a minimal surface (see abstract of the author's second paper, below) is discussed.

* See preceding abstract.

23. Mr. G. Y. Rainich: *The analytic function on a minimal surface.*

The Codazzi equations, to which the tensor usually represented by the second differential form is subjected, have in the case of a minimal surface the form of the Cauchy-Riemann equations which are satisfied by an analytic function. To make the analogy complete, we have to consider an analytic function as defining in each point of the plane a linear vector transformation. The Cauchy integral theorem is immediately applicable to the function on a minimal surface, and this leads to the definition of a residue. The Gaussian curvature at each point is equal to minus the square of the modulus of the function at that point. If a surface is given by its first form and the coefficients satisfy a certain differential equation, the analytic function is defined to a constant of integration, the values of which correspond to associated surfaces. Certain of the results are extended to surfaces of constant mean curvature.

24. Professor Dunham Jackson: *On the gradient of a functional.* Preliminary communication.

This paper is concerned with a definition of the directional derivative of a functional, the determination of the direction in function space for which the value of this directional derivative is a maximum, and the evaluation of the derivative in any other direction as the product of the above-mentioned maximum value by the cosine of the angle between the two directions in question.

25. Dr. T. H. Gronwall: *Extension of a theorem due to Lerch.*

Lerch has shown that when $f(x)$ is continuous for $0<x<1$ and $\int_0^1 x^n f(x)dx = 0$ for $n = 0, 1, 2, \cdots$, then $f(x)$ vanishes identically. In the present note it is shown that when $f(x)$ is absolutely integrable and the above equation holds, $f(x)$ vanishes identically on any sub-interval where it is continuous.

26. Dr. T. H. Gronwall: *On the Cesàro summability of Fourier's and Laplace's series.*

Fejér has shown that any Cesàro sum of order one of the Fourier series of a continuous function takes only

values between the extreme values of the function, and also that the same property holds for the sums of order two of the Laplace series. Considering also fractional values of the order k of the sums, the present paper proves that $k = 1$ and $k = 2$, respectively, are the smallest values of k with the above property.

27. Professor J. R. Kline: *A note concerning closed non-dense linear sets which are enumerable.*

In both the first and the second editions of Hobson's *Theory of the Functions of a Real Variable*, the author attempts to prove that a non-dense closed set is enumerable if its complementary intervals are such that every one abuts on another at each of its ends. Professor R. L. Moore gave an example of a set satisfying the conditions of the hypothesis of Hobson's theorem, which is not enumerable. In the present paper it is shown that in case it is stipulated that the non-dense closed set and all its derived sets have the property of having their complementary intervals each abut on another at each of its ends, then the set is enumerable.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The forty-fourth regular meeting of the San Francisco Section was held at the University of California on October 25, 1924. In the absence of the Chairman of the Section, Professor Carpenter, the meeting was called to order by the Secretary of the Section. Professor Hedrick was elected temporary chairman. The total attendance was thirty, including the following twenty-five members of the Society:

Alderton, Allardice, Andrews, Bell, Bernstein, Blichfeldt, Growe, Haskell, E. R. Hedrick, Hotelling, Irwin, Lehmer, Sophia Levy, McCarty, McFarland, F. R. Morris, Moreno, Noble, Pehrson, T. M. Putnam, Robertson, Schmiedel, Pauline Sperry, A. R. Williams, Wong.

The following officers were elected for the coming year: Chairman, Professor E. R. Hedrick; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, H. F. Blichfeldt, B. A. Bernstein.

It was decided to hold regular meetings of the Section during 1925 as follows: at Stanford University on April 4, at the University of Oregon on June 19, and at the University of California in the fall, the exact dates and places to be determined by the officers of the Section.

Titles and abstracts of papers read at this meeting follow. In the absence of the author, Professor Bateman's paper was read by Mr. H. P. Robertson. Mr. Reimer was introduced by Professor Bernstein.

1. Professor Harry Bateman: *Numerical solution of an integral equation.*

Hafen has shown that the distribution of electricity on the circular plates of a parallel plate condenser can be found by solving a linear integral equation of the second kind. In the present paper the author solves this equation numerically by a method which is in some respects analogous to the method of least squares, and discusses the method in a general way, remarking on the question of convergence.

2. Professor E. T. Bell: *An algebra with singular zero.*

This algebra, which is that of the symbolic calculus for dealing with relations between the individual terms of sequences, is abstractly identical with common algebra, when properly interpreted, except that zero, defined by $a-a=0$, where a is any element of the algebra, has singular properties, such as $b+a-a\neq b$ (b an element of the algebra), $b(a-a)\neq 0$. An outline is given of the applications of this powerful tool for investigating the properties of numbers defined by sequences, also of the Bessel coefficients.

3. Professor E. T. Bell: *On certain functions of two variables and their integrals related to the Bessel coefficients.*

The functions $L_n(\lambda, \mu)$ in question are the coefficients

of $t^n/n!$ ($n = 0, \pm 1, \pm 2, \cdots$) in the Laurent expansion, absolutely convergent for $|t| \neq 0$, of exp $(\lambda t + \mu t^{-1})$. By taking the coefficients of $t^n/n!$, and not of t^n, it is feasible to apply the umbral calculus of Blissard to the study of these functions and their integrals. For $\lambda = -\mu = \frac{1}{2}z$ the functions become the Bessel coefficients $J_n(z)$. Thus the relations between the $J_n(z)$ and the integrals dependent on them are obtained here in special cases. The L_n are readily expressed in terms of the J_n for irrational arguments. Many of the integrals thus evaluated in terms of Bessel coefficients appear to be new. The symbolic method applies equally well to the functions defined by exp $(F(t))$, where $F(t)$ is any finite or infinite polynomial in t, t^{-1}. These also are expressible in terms of Bessel coefficients. The use of the symbolic method cuts the computations to a minimum.

4. Professor E. T. Bell: *Modular equations and quadratic forms.*

The elliptic modular equation associated with the transformation of order n implies and is implied by each of several elementary arithmetical theorems concerning the representations of integers in certain quadratic forms with integer coefficients, of which at least one is divisible by n. For $n = 7$, in many respects the simplest case, there are 256 associated theorems, of which 4 refer to binary forms, 56 to ternary, and 196 to quaternary. These are given in full, in a condensed form.

5. Professor E. T. Bell: *On generalizations of the Bernoullian functions and numbers.*

All generalizations are segregated into two classes, (1) those obtained by modifications of the difference equations of definition, (2) those derived from modifications of the symbolic equations of definition. An example of (1) is Nörlund's memoir in ACTA MATHEMATICA, vol. 43, pp. 121–196. Here (2) only is discussed and illustrated by an extensive generalization of the "ultra-Bernoullian functions" of Krause (LEIPZIGER BERICHTE, vol. 55, pp. 39–63) and Appell (ARCHIV FÜR MATHEMATIK UND PHYSIK, vol. 4 (1902), pp. 292–5). The number of attainable generalizations is infinite. Suggestions are given for classifying this infinity.

6. Professor E. T. Bell: *Transformations of relations between numerical functions.*

From consideration of functions whose arguments are the elements of an abelian semi group, Cesàro (ANNALI DI MATEMATICA, (2), vol. 13, p. 155, (12), (13)) obtained a remarkable inversion of series, "le plus général que l'on connaisse," and states that the well known Möbius inversion is a special case. It is here shown that either one of the Cesàro or Möbius inversions implies the other, and that both are contained in an extremely general theorem of reciprocity, of which the Cesàro-Möbius inversion is the simplest example possible.

7. Professor E. T. Bell: *A new type of class number relations.*

The novelty is that the sums of class number functions are expressible as functions of the real divisors alone of a single integer when and only when the fixed integer in the arguments of the class number functions is a square. This paper appears in the October, 1924, number of the TRANSACTIONS.

8. Professor B. A. Bernstein: *A general theory of representation of finite operations and relations.*

At the International Mathematical Congress at Toronto, the author presented a method of representing arithmetically any abstract binary operation and any dyadic relation in a finite class. The method is now extended to the case of operations other than binary and relations other than dyadic. The representations bring out clearly the nature of an operation or relation.

9. Professor Florian Cajori: *Fanciful hypotheses on the origin of the forms of the Arabic numerals.*

The author describes numerous attempts, which have been made during the past nine centuries, to explain the origin of the forms of our nine numerals and the zero from the composition of short lines, or of angles, or of dots, and points out that all the hypotheses advanced proved fruitless because they failed to coordinate the known facts relating to the forms of the numerals at different times and in different countries, and failed to suggest useful new lines of further research.

10. Professor M. W. Haskell: *Note on the self-dual quintics.*

At the Mathematical Congress at Toronto, the author developed a method for deriving an unlimited number of curves which are autopolar with respect to a finite number of conics. In the present paper he exhibits two such curves of the fifth order, viz. (*a*) The curve whose equation in polar coordinates is $8\varrho^5 \cos 5\varphi - 15\varrho^4 + 10\varrho^2 - 3 = 0$ is a quintic with five cusps, invariant under a dihedral group of 10 linear substitutions, and hence autopolar with respect to six conics; (*b*) The curve whose equation is $(9\varrho^5 - 5\varrho^5) \cos 3\varphi - 15\varrho^4 + 15\varrho^2 - 4 = 0$ is a quintic with three cusps and three double points, invariant under a dihedral group of 6 linear substitutions, and hence autopolar with respect to four conics.

11. Professors E. R. Hedrick and Louis Ingold: *On certain attempted extensions of the theory of functions to three dimensions.*

The authors discuss attempts made by themselves and by others to extend the ordinary theory of functions of a complex variable to three dimensions by means of definitions of the product or of the quotient of two three-dimensional vectors. The nature of the difficulties encountered and the reasons for them are pointed out. Another road toward extension of the theory was finally adopted, and has been presented to the Society (see this BULLETIN, vol. 29, p. 437), but the reasons for abandoning the more obvious path have not been stated, and it seems not without interest to state them.

12. Dr. Harold Hotelling: *Three-dimensional manifolds of states of motion.*

To every two-dimensional dynamical problem with constant total energy there corresponds a three-dimensional manifold, two coordinates determining the position of the moving particle on the surface given by the problem, and the third the direction of its motion. The topology of this manifold, which is important in the study of periodic orbits, depends only upon that of the portion of the given surface on which the particle may move. Characterizations of all of these manifolds and formulas for their principal invariants are given.

13. Dr. F. R. Morris: *Development of annuity formulas without series.*

The formula $s = (1+r)^n$ may be taken as the definition of the amount on one unit with interest at rate r per period for n periods. Then $(1+r)^n-1$ is the interest on one unit. $(1+r)^n-1$ is also the value of an annuity with periodic rent r. The author makes use of these facts to derive the well known annuity formulas without the help of series.

14. Mr. E. H. Reimer: *Application of Weyl's geometry to differential geometry in two dimensions.* Preliminary report.

If neighboring vectors of zero length are equivalent, the geometries of Weyl and Eddington coincide for the surface $S(u_1, u_2)$. For Eddington's $\lambda g_{ij} = R_{ij}$, the Gauss equations give $\lambda = R$. Either of two vectors $V_i(u_1, u_2)$ $(i = 1, 2)$ in the asymptotic directions of length $\sqrt{R}$ gives a Weyl vector. Accents denoting covariant derivatives, the Codazzi equations become $V_1' = \pm V_2'$, according as the parameters are (u_1, u_2) or (u_2, u_1).

15. Mr. H. P. Robertson: *On certain solutions of the cosmological equations.*

This paper concerns itself with certain solutions of Einstein's equations in a space-time which may be described by orthogonal coordinates. A solution is found which gives the field due to a line or plane distribution of matter and electricity. It is proved that certain types of line element lead only to the central-symmetric field of Schwarzschild. Other solutions are found which give rise to theorems of interest in differential geometry and the new physics; in these cases simple data of observation are the criteria of the form of space-time.

B. A. BERNSTEIN,
Secretary of the Section.

THE SEVENTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The seventeenth regular meeting of the Southwestern Section of this Society was held at Iowa State College on Saturday, November 29, 1924. The following thirty members of the Society were present at the meeting:

E. S. Allen, Ashton, Birkhoff, Brenke, Chittenden, Coffin, J. T. Colpitts, Daniells, C. W. Emmons, Gaba, Gouwens, Herr, Louis Ingold, J. V. McKelvey, Martha McKelvey, Pattengill, Pierce, J. F. Reilly, H. L. Rietz, M. M. Roberts, Roever, Rusk, E. R. Smith, Helen Smith, Snedecor, Stouffer, Tappan, J. S. Turner, W. H. Wilson, Roscoe Woods.

Professor E. R. Smith occupied the chair, being relieved during the afternoon session by Professor Birkhoff. The morning session and part of the afternoon session were devoted to the reading of the papers listed below.

During the afternoon session Professor William H. Roever gave a special lecture by invitation of the program committee on *Some phases of descriptive geometry*. This paper will appear in full in an early issue of this BULLETIN.

It was voted to hold the next meeting of the Southwestern Section at Kansas City, Missouri, in conjunction with the meeting of the American Association for the Advancement of Science. The following program committee was elected: Professors R. L. Moore (chairman), Louis Ingold, and E. B. Stouffer (secretary).

The titles and abstracts of the papers read are given below. In the absence of the authors the papers by Dr. Green and Professor Moore were read by title.

1. Professor G. D. Birkhoff: *On the structure of matter.*

The author developes a theory of matter, and electricity attached to matter, on the assumption that the matter is a perfect adiabatic fluid in which every disturbance travels with the velocity of light.

2. Professor E. W. Chittenden: *Note on a compact set which is not perfectly compact.*

R. L. Moore has called attention to the fact that the class of transfinite ordinals is compact but not perfectly compact. The present note contains an example of an enumerable set which is compact but not perfectly compact.

3. Professor W. H. Wilson: *On two general functional equations.*

Several special cases of the functional equations

$$f(x+y)+g(x-y) = h(x)\,k(y),$$

and

$$F(x+y)\,G(x-y) = H(x)+K(y),$$

where x and y are independent, have been considered. The author assumes no relationships between the functions and places no restrictions on them. He expresses these eight functions directly in terms of the functions which satisfy the Cauchy equations $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(x+y) = \varphi(x)\,\varphi(y)$. By virtue of the results obtained in this paper, any information concerning these basic Cauchy equations may be applied immediately to the general equations proposed above or to special cases of these general equations.

4. Professor W. H. Wilson: *On a certain functional equation.*

The author discusses the functional equation

$$f(x+y)-f(x-y) = yg_k(x)+y^3g_{k-1}(x)+\cdots+y^{2k-1}g_1(x),$$

in which x and y are independent (not necessarily real), and $f(x)$, $g_1(x)$, $g_2(x)$, $\cdots$, $g_k(x)$ are functions to be determined. It is shown that $g_i(x)$, for each value $i = 1, 2, \cdots, k$, satisfies the equation

$$\text{(A)} \qquad \sum_{j=0}^{2i}(-1)^j\,{}_{2i}C_j\,g_i[x+(2i-j)y] = 0,$$

where ${}_{2i}C_j$ is the coefficient of a^j in the expansion of $(1+a)^{2i}$. In addition to this it is obvious that

$$f(2x) = f(0)+xg_k(x)+x^3g_{k-1}(x)+\cdots+x^{2k-1}g_1(x).$$

This separation of the unknown functions is effected without restricting the functions (as to continuity, etc.). Equa-

tions of form (A) have been discussed by the writer (AMERICAN JOURNAL, vol. 40 (1918), pp. 263 et seq.), in the light of various restrictions on the functions $g_i(x)$.

5. Professor J. S. Turner: *The spherical analogue of central forces.*

If a particle is constrained to move on a sphere, the projection of its acceleration at P upon the tangent plane at P is here called its tangential acceleration. The line of action of the tangential acceleration touches a certain great circle; if O is a point on this great circle, the tangential acceleration at P is said to be directed to O. A tangential acceleration which is always directed to a fixed point O is called a central tangential acceleration. It is proved that if a particle moving with a central tangential acceleration directed to O describes a curve on the sphere, then $v \sin p$ is constant, where v is the velocity at P, and p is the arcual perpendicular from O on the arcual tangent to the curve at P. It is also proved that if the particle describes a spherical conic under a central tangential acceleration directed to a focus S, this acceleration is inversely proportional to $\sin^2 SP$; and a simple expression is found for the velocity at P.

6. Professor J. S. Turner: *An extension of the property of the circle known as "Simson's line".*

Let ABC be a triangle inscribed in a circle, and let P, Q be two inverse points with respect to the circle. Take L, M, N, the images (symmetric points) of P in BC, CA, AB, and join QL, QM, QN, cutting BC, CA, AB in X, Y, Z respectively. Then the points X, Y, Z are collinear.

7. Professor Julia T. Colpitts: *Entire functions defined by series of the form* $f(z) = \sum f(n)(z^n/n!)$.

In the first part of the paper, $f(n)$ is taken as a polynomial in n. It is proved that the necessary and sufficient condition that a function of genus one have a finite number of zeros is that $f(n)$ be the product of α^n(α constant) and a polynomial in n. In the second part, $f(n) = 1/(n+a)^s$, $s > 0$ and rational, and a any real number except a negative integer or zero. The nature of the zeros and some approximations to their values are determined.

8. Professor T. A. Pierce: *An explicit solution of a congruence.*

A. Hurwitz employed an interpolatory function (mod p) in finding the number of real roots of a congruence. Professor Dickson used the same function in constructing a theory of modular invariants. The author shows how the interpolatory function may be used to furnish an explicit solution of a congruence whose modulus is a prime.

9. Dr. C. F. Green: *On the summability and regions of summability of a general class of series of the form* $\sum c_n g(x+n)$.

The first part of the paper is devoted to the determination of the character of the regions of summability of the series for Cesàro type of summability of order r. The regions are half planes bounded by straight lines whose abscissas are dependent upon the real part of the exponents in the asymptotic representation of the series. A certain class of auxiliary series (a generalized Dirichlet series) which possesses the same regions of summability as the above series is used as an aid in the determination of the abscissas of summability. Relations between the regions of summability are discussed for the general series and also for particular series of the above type.

10. Professor Louis Ingold: *Extensions of the Gauss and Codazzi relations.*

Relations connecting certain differential parameters of the first and second orders are obtained which are analogous to the well known relations of Gauss and Codazzi connecting the first and second fundamental quantities of surface theory. The extension to n dimensions is also given.

11. Professor E. S. Allen: *The game of "bell and hammer".*

The game of "bell and hammer" is begun by the "sale" of five cards at auction. M. Jequier of Neuchatel investigated the probable value of these cards, neglecting the last phase of the game; the present paper completes the investigation. Among other questions, that of the probable number of throws needed to obtain a certain sum exactly (either by one throw or by a succession) is studied.

12. Professor R. L. Moore: *Concerning upper semi-continuous collections of continua which do not separate a given continuum.*

A collection of continua G is defined to be an *upper semi-continuous* collection provided it is true that, if p is a continuum of the collection G and $p_1, p_2, p_3, \cdots$ is a sequence of continua of the collection G such that the lower distance from p_n to p approaches zero as a limit as n approaches infinity, then so does the upper distance from p_n to p. It is shown that if, in a plane S, M is a bounded continuum which does not separate S and G is a countable upper semi-continuous collection of subcontinua of M, no one of which separates M, then the sum of all the continua of the set G does not separate M.

13. Professor R. L. Moore: *Concerning sets of segments which cover a point set in the Vitali sense.*

In volume 5 of FUNDAMENTA MATHEMATICAE, J. Splawa Neyman showed that if, in space of one dimension, a closed point set K of measure zero is covered in the Vitali sense by a set of segments G then, for every position number e, G contains a subset G_e such that G_e covers K and such that the sum of the lengths of the segments of G_e is less than e. Sierpinski has raised the question whether this theorem remains true if the requirement that K be closed is omitted. This question here is answered in the negative. But if the requirement in question is replaced by the requirement that there do not exist infinitely many segments of the set G, all of length greater than the same positive number, then the resulting proposition is true.

14. Professor F. Reilly: *On Lidstone's demonstration of the osculatory interpolation formula.*

The author shows that the osculatory interpolation formula of Lidstone can be generalized so that it will involve finite differences of order $2h+1$, and its graph will have contact of order k with the partial interpolation curves, in a manner analogous to that previously employed in generalizing the formula as demonstrated by Sprague. The two generalized formulas are, in general, not identical.

E. B. STOUFFER,
Secretary of the Section.

REMARKS ON THE FOUNDATIONS OF GEOMETRY*

BY OSWALD VEBLEN

1. *Relation between Matter and Space.* The foundations of geometry must be studied both as a branch of physics and as a branch of mathematics. From the point of view of physics we ask what information is given by experience and observation as to the nature of space and time. From the point of view of mathematics, we ask how this information can be formulated and what logical conclusions can be drawn from it.

It is from the side of physics that has come the most important contribution in the last two decades. The experimental physicists and the astronomers have uncovered facts which thus far can only be reconciled with one another by the method due to Einstein. This method consists essentially in regarding the events of space-time as a four-dimensional manifold in which there is a Riemann metric. The method is radically different in principle from that of the classical geometry and mechanics. Yet the quantitative results which flow from it are so closely in agreement with those of the older theory that they can be distinguished experimentally only in a few cases and with great difficulty.

To understand the exact way in which the euclidean geometry and the Newtonian mechanics can be regarded as a first order approximation to the Einstein theory of gravitation seems to me to be of great importance in understanding the physical significance of geometry altogether. For it requires us to be clear as to what we mean physically by straight lines, axes of inertia, absolute rotation,

* Presidential address delivered before the Society, December 31, 1924.

and so on. Therefore I propose, with your kind indulgence, to make my first remarks on this subject. It seems to me that it is a suitable question to discuss in this sort of an address, in which one is not expected to pile new bricks on the edifice of science but rather to make a few observations as to how the brick-layers are getting on with their work.

In the Newtonian theory of gravitation the motion of a particle is determined by the interplay of two forces, the attraction (according to the inverse square law) of all other particles, and the particle's own inertia, i. e., its tendency to move uniformly in a straight line. The inertia of a particle is an unexplained and unchanging property of the particle itself, although it clearly requires a euclidean geometry and a theory of time to state what is meant by the phrase "uniformly in a straight line".

The lack of a physical basis for inertia was seen by Mach, if not by others before him, as a weakness of the Newtonian mechanics, and Mach put forward the idea that the inertia of a particle as well as the gravitational attraction on it is determined by all the rest of the matter in space. This conception was very influential in developing Einstein's gravitation theory and is taken account of in this theory by merging the ideas of inertia and gravitational attraction into a new idea. The motion of a particle is completely described by saying that its world-line is a geodesic of the differential form

$$ds^2 = g_{ij}dx^i dx^j. \tag{1}$$

The effect of the rest of the matter in the universe is to determine the values of the functions g_{ij}, and the functions g_{ij} in turn determine the motion of the particle.

This motion can be described in terms of inertia, gravitation, etc., if we employ a properly chosen euclidean metric as an approximation to the Riemann metric of the differential form (1). The approximation may be described as follows.

Consider an event in our galaxy well separated in space and time from any large mass of matter. A set of coordinates with this event as origin may be chosen by a well known method (Riemann normal coordinates) so that (1) becomes

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2}$$

at the origin, and the equations of the geodesics through the origin become

$$x = as, \quad y = bs, \quad z = cs. \tag{3}$$

Since there is no mass of matter near the origin it follows that the formula (2) which holds rigorously only at the origin is also valid to a high degree of approximation for a large region R. In this region the geodesics which do not pass through the origin, as well as those which do, will be represented to a high degree of accuracy by first degree equations.

In other words special relativity holds with high accuracy throughout R. Any two coordinate systems with regard to which the conditions (2) and (3) hold good are related by a Lorentz transformation and are said to be in uniform motion, the one with regard to the other. The three-dimensional locus of a moving particle is the same as the three-dimensional locus of a ray of light. This is the physical content of the statement that a particle moves along a straight line.

For regions in the immediate neighborhood of masses of matter such as the sun or the earth, it is not true that the equations of geodesics are linear. But this does not prevent us from using the coordinate system (x, y, z, t) all around, and all through, the earth and sun. In this region as well as in the region R a straight line is a curve whose equation is of the first degree with regard to the coordinates x, y, z.

The coordinate system was undetermined up to a Lorentz transformation. We now determine it uniquely by the condition that it shall be at rest with respect to the sun.

This particular coordinate system we call an inertial frame of reference F. For small velocities, the Lorentz transformation is very closely approximated by the transformation

$$(4)\qquad \begin{cases} \bar{x} = a_{11}x + a_{12}y + a_{13}z + c_1t + d_1, \\ \bar{y} = a_{21}x + a_{22}y + a_{23}z + c_2t + d_2, \\ \bar{z} = a_{31}x + a_{32}y + a_{33}z + c_3t + d_3, \end{cases}$$

in which the coefficients are all constants and the numbers a_{ij} form an orthogonal matrix. Any frame of reference which arises from F by a transformation of this sort is also called an inertial frame.

With respect to an inertial frame the path of light is still very approximately linear even close to the earth or the sun, but the path of a material particle is more nearly a second degree curve. This divergence of the particle paths from the light paths is accounted for by introducing the concept of force and the Newtonian mechanics. This mechanics says that in the absence of forces, particles would move uniformly in straight lines—i. e., in light paths. But the matter composing the earth brings an attracting force to bear on any particle and consequently the actual path diverges from that of light.

Thus the Newtonian mechanics is to be regarded as a device for separating out the local influence of matter (using an astronomal scale for the term "local") from the influence of matter as a whole. All the matter in the universe determines what lines are straight, what pairs of lines are perpendicular or parallel, what triangles are similar to each other, and so on. It also determines whether any given type of motion has or does not have acceleration; for example, it determines the plane in which a Foucault pendulum swings and thus gives an absolute magnitude for the rotation of the earth. On the other hand the local influence of the earth, the sun, and the planets is taken care of by means of forces which show how the actual paths of particles are diverted from what they would be without the presence of these bodies.

Consider the state of affairs inside a spherical shell of matter. The Newtonian theory assures us that the shell exerts no force of attraction on a particle inside it, a statement which must have seemed paradoxical to all of us so long as we took it to mean that the shell exerts no influence on the particle. We now see that the shell has a share in determining the very geometry according to which the particle moves, and that if the shell is set into rotary motion, say with regard to another shell which contains it, there must be a tendency to rotate the axes of inertia and thus to modify the whole geometry of the interior.

A related question might conceivably arise in astronomy. Suppose we have a very large but very sparsely distributed mass of matter which is far away from other masses. It is conceivable that such a mass might have a certain degree of autonomy in the determination of its axes of inertia. To see this, let us suppose an observer to start somewhere in the interstellar space of our galactic system and to travel without ever coming close to a large mass of matter until he arrives somewhere in the interstellar or intermolecular space of the distant mass. Anywhere he stops and sets up a coordinate system of the type we have been considering, he will find that he has, to good approximation, an inertial system extending over a quite extensive region. But as he travels, the functions g_{ij}, expressed in the original coordinate system, are gradually changing and therefore the relation between the original coordinate system and the inertial system is changing. The relation between his first inertial system and his last one will not in general be capable of representation by a Lorentz transformation. Hence the one system will appear to be accelerated with respect to the other. Such an acceleration would be expected to show itself most obviously in one or all of three ways: in a linear acceleration of the mass from or toward us, in an acceleration toward or from its center, in a rotation of its axes of inertia relative to ours.

There is good reason to believe that the domain of our axes of inertia reaches well out toward the limits of the visible universe. For the velocities of the stars are small and there is nothing known about their motion which conflicts with a Newtonian mechanics based on our inertial axes. It is not impossible, however, that the spiral nebulae might be masses of the sort to which we are trying to attribute partially autonomous inertia. They are well isolated, enormously large, and enormously distant, as recent work by Hubble establishes more clearly than ever before. Moreover their radial velocities relative to the sun are very large, they have an expansive motion relative to their own centers and they have large rotary motion. Indeed, measurements on the internal motions of certain spirals have been made by Van Maanen which Jeans found he could explain by Newtonian mechanics relative to the usual axes only if he assumed a law of attraction different from the inverse square law. Of this law Jeans says "The general effect of the modified gravitational force appears to be one of seizing the particle and compelling it to revolve about the nucleus of the nebula with an angular velocity which approximates to that of the nucleus itself rather than to that demanded by the conservation of angular momentum."

Referring to the work of Jeans, E. W. Brown asked me whether there might not be a possible relativity explanation of the apparent rotation of axes of reference of the nebula. My reply was that the relativity theory provides a physical reason for expecting this rotation, provided the nebula is sufficiently large to have a degree of autonomy with respect to inertia. In this case we should expect to describe the internal motions of the nebula in terms of a Newtonian mechanics stated in terms of inertial axes different from ours.

I am citing this highly speculative possibility in the hope that it will serve to make the problem of the foundations of geometry more vivid from a physical point of view. It

may also serve to indicate some of the elements which must be thought of in a non-static solution of Einstein's gravitational equations. In order to take account of the distribution of matter throughout space in determining the values of the functions g_{ij} at any point, it would seem that we ought to have integral equations to be satisfied by the g's. But whether the partial differential equations given by Einstein are compatible with such integral equations, I do not know.

2. *The Geometry of Paths.* The studies of Riemann geometry which have been stimulated by the relativity theory have revived interest in the foundations of differential geometry. Here one starts with the observation that when arbitrary coordinates x undergo a general analytic transformation the differentials of these coordinates suffer only a linear transformation.

The differentials at a point may be regarded as representing directions emanating from this point. Thus we have at each point a projective, and also an infinity of affine and euclidean geometries to represent the relations among these directions.

But to begin with there is nothing to connect the differentials at any point x with the differentials at a point $x+dx$. Such a connection must be obtained by relations involving the higher differentials of the coordinates. It is accomplished in one manner by the introduction of a Riemann metric dependent on a quadratic differential form (1) which makes the process of taking higher differentials definite. A clear view of the Riemann geometry from this standpoint was first obtained by means of Levi-Civita's conception of infinitesimal parallelism. This amounted to finding a geometrical interpretation for the Christoffel symbols

$$\left\{ \begin{matrix} ij \\ k \end{matrix} \right\}$$

as the coefficients of a transformation which carries the

components of a vector at a point x into the components of a uniquely determined vector at any point $x+dx$. It provides a definite way to visualize the limiting process by which we carry over the conceptions of euclidean geometry in a finite domain to the Riemann geometry in an infinitesimal domain.

Levi-Civita's construction was obtained by regarding the Riemann manifold as immersed in a euclidean space. The euclidean space is not essential and Weyl defined infinitesimal parallelism as an intrinsic property of the manifold itself. He also generalized it so that

$$V^i = \Gamma^i_{\alpha\beta} V^\alpha dx^\beta$$

represents an arbitrary affine transformation from x to $x+dx$. He thus arrived at the conception of a general affine connection determined by arbitrary functions $\Gamma^i_{\alpha\beta}$ which are symmetric in the lower indices.

A further step in generalization was taken by Cartan. Let us imagine that with each point x of the manifold which we are talking about there is associated a projective space S, one point of the projective space being identical with x itself. Then if we can specify a definite projective transformation from S to the projective space associated with each point $x+dx$, and do so by means of a set of functions of x, we have a projective connection in the sense of Cartan.

All these methods of connecting up the differentials at one point with the differentials at the points in its neighborhood can be seen very clearly in terms of the geometry of paths. By a geometry of paths I mean the theory of any system of analytic curves in an n-dimensional manifold, the curves being such that any point is joined to any other point in a sufficiently near neighborhood of itself by one and only one curve of the system. Such a system of curves is the natural generalization of the straight lines of euclidean geometry and of the geodesics of Riemann geometry.

In a very general case the paths can be represented as solutions

$$x^i = f^i(s) \tag{5}$$

of a set of differential equations,

$$\frac{d^2x^i}{ds^2} = \Gamma^i\left(x, \frac{dx}{ds}\right) \tag{6}$$

in which the right hand members are functions of the x's and their first derivatives. The invariant theory of these differential equations under transformations of the coordinates x is the affine geometry of paths. This geometry can be developed in some detail in case the functions Γ^i are homogeneous of degree 2 in dx/ds. Its most powerful method consists in transforming (6) to normal coordinates, that is, to coordinates such that

$$y^i = \xi^i s, \tag{7}$$

with ξ^i constant, represent the paths through the origin. Whenever the coordinates x undergo an arbitrary analytic transformation the normal coordinates receive merely a linear transformation with constant coefficients.

If we define a system of straight lines by means of linear equations with respect to normal coordinates, we obtain an associated euclidean space in which the straight lines through the origin coincide with the paths (7) through it. Since an arbitrary transformation of the coordinates x brings about a linear transformation of the normal coordinates it brings about an affine transformation of the associated euclidean space.

The choice of the differential equations (6) to represent the paths has determined the type of the parameter representation (5) to a certain extent and has also introduced a relation between the parameters representing different paths through the same point. If we subject the parameter in (5) to an arbitrary transformation the equations of the paths will, in general, no longer satisfy differential equations of the form (6). But the differential equations of the paths

can be written in a form

$$\frac{\dfrac{d^2x^i}{ds^2}-\Gamma^i\left(x,\dfrac{dx}{ds}\right)}{\dfrac{dx^i}{ds}}=\frac{\dfrac{d^2x^j}{ds^2}-\Gamma^j\left(x,\dfrac{dx}{ds}\right)}{\dfrac{dx^j}{ds}} \tag{8}$$

which is unaltered by transformations of the parameter s.

A transformation of the parameter brings about, in the euclidean space associated with each point by means of the normal coordinates, a transformation which is not in general affine. It thus requires the introduction of points at infinity in each of these euclidean spaces. In certain cases the adjunction of the points at infinity will convert the euclidean spaces into projective spaces.

If we specialize the functions Γ^i so that (6) takes the form

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0 \tag{9}$$

where $\Gamma^i_{\alpha\beta}$ are functions of x, we have the sort of affine geometry of paths which has been studied by Eisenhart, Thomas, and myself and which is equivalent to the affine geometry of Weyl. For infinitesimal parallelism is simply invariance of the components of a vector in the neighborhood of the origin of normal coordinates and covariant differentiation is merely differentiation with respect to normal coordinates at the origin. Indeed, differentiation of any order with respect to these coordinates at the origin is an invariant operation which enables us to find a complete set of invariants for the affine geometry.

When the parameter s in the equations of a path satisfying (9) undergoes a transformation which is not linear, the functions $f^i(s)$ in the equations of the path (5) are transformed into functions which are solutions of equations

$$\frac{d^2x^i}{ds^2}+\Lambda^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0,$$

in which

$$\Lambda^i_{\alpha\beta}=\Gamma^i_{\alpha\beta}+\delta^i_\alpha\varphi_\beta+\delta^i_\beta\varphi_\alpha,$$

the functions φ_α being components of a vector, and δ^i_j the Kronecker delta. In a very general case it is found that the normal coordinates undergo a linear fractional transformation when the functions Γ are changed in this manner into the functions A. We thus find that a set of homogeneous coordinates whose ratios are equal to normal coordinates for the functions containing the indeterminate vector φ are suitable for the determination of projective invariants. These projective invariants can be said to give genuine properties of the paths since they are independent both of the choice of the coordinates x and of the particular parameter representation of the paths.

Certain choices of the vector φ bring about particular transformations of the parameters s in the equations (5) which amount to non-affine projective transformations of the euclidean space mentioned above. By adjoining points at infinity to this space we obtain a projective space which is essentially the same as that considered by Cartan. The projective transformations determined by φ bring about a class of transformations of the parameters of all paths through the origin of normal coordinates. It is an interesting fact that any transformations of the parameter on any path may be obtained in this way.

The successive specializations which we have been introducing in the specification of the system of paths can all be regarded as closer and closer restrictions on the manner in which the paths are simultaneously given a parameter representation. This process can be continued by assuming that the equations (9) possess a first integral, say a homogeneous quadratic first integral

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{constant}.$$

This enables us to define the parameter along any path by means of the definite integral

$$\int\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt$$

and gives the parameter many of the characteristic properties of distance. Finally by requiring that the path actually minimize this integral we specialize down to a Riemann geometry.

In thus shadowing forth something like a set of axioms for the various types of path geometry, I have not given the calculus of variations its due. The problem of determining those systems of paths which can be regarded as extremals of a definite integral is one whose solution will doubtless modify our views on the foundations of differential geometry.

3. *Choice of Undefined Terms.* In studying the Riemann geometry and its generalizations we make use of the euclidean geometry at every turn. Hence our interest in these subjects by no means kills our interest in the foundations of euclidean geometry. But these studies and the revival of the physical point of view necessarily have an influence on our preference among the many ways of formulating the axioms.

The Riemann geometry and its generalizations all start with the assumption that the set of points with which they deal is a portion of an n-dimensional manifold, or an n-cell in the sense of analysis situs. This assumption may be formulated by asserting that there exists a way of labelling the points by means of ordered sets of n numbers

$$(x^1, x^2, \cdots, x^n),$$

one set of numbers for each point and one point for each set of numbers. This is equivalent to assuming that an n-cell is a set of points capable of supporting a euclidean geometry. For if there is a coordinate system there is a system of curves which are linear with respect to it, and which function as the straight lines of a euclidean space; and if the points are in correspondence with a euclidean space, a coordinate system can be set up. Thus any set of postulates for a euclidean space gives rise to a set of postulates for an n-cell.

Of course, a set of postulates for any particular one of the generalized geometries which we are considering would characterize an n-cell just as well as the postulates for euclidean geometry. Any two of these geometries are in a relation which may be stated as follows: If A be an arbitrary point of the space described by one n-dimensional geometry, A is surrounded by an n-dimensional region R_A in which the points are uniquely denoted by coordinates x and in which the relations of the geometry (such as the formula for distance or the equations of the paths) are expressed by analytic functions in terms of these coordinates. If B be an arbitrary point of the space described by any other n-dimensional geometry there is a region R_B having analogous properties for this geometry, and such that between the coordinates y of R_B and the coordinates x of R_A there is an analytic relation

$$y^i = f^i(x)$$

with a unique inverse.

Thus in a certain sense the two geometries are analytically related. It is not that there is an analytic transformation of the one geometry into the other. But the fundamental loci of each geometry determine a class of coordinate systems in terms of which these loci are analytically representable, and there exists an analytic transformation of any coordinate system of the one geometry into any coordinate system of the other.

One of these geometries is that of Euclid. Hence a perfectly sound way of stating the postulates of any of the geometries we are considering is first to give a set of postulates for euclidean geometry and then to postulate that the geometry which we are describing is analytically related to that of Euclid.

It seems clear to me that when they are to be used for the purpose of characterizing an n-cell we can definitely prefer certain sets of postulates of elementary geometry to others. The purpose is to characterize a set of points; therefore there should be as few other elements as possible.

For example, a set of postulates in terms of points, lines, planes and congruence is less desirable than one in terms of points and lines only. Moreover a set of postulates about points and lines in which the lines are regarded as sets of points is to be preferred to one in which the lines are objects of some other sort to which the points are joined by means of an undefined relation.

Between a set of postulates in terms of points and lines and one in terms of points and the betweenness relation, there would doubtless be little to choose. In either case, space is characterized as a class of points which are arranged in sub-classes related among themselves in a particular way.

The fact that physical discussions seem always to treat the line and plane as sets of points would seem to confirm this preference. Also the fact that in its direct application to nature the euclidean geometry is used to give an approximate description of a portion of the universe would seem to establish a presumption in favor of axioms which are stated for only a finite region of space.

I do not mean to suggest that we should try to reduce the number of undefined terms to a minimum. For in a mathematical science which has physical applications there is much to be said for making an undefined term out of each object or relation which has an independent physical definition.

A physical definition is made in terms of human operations. Thus, for example, to define what a point is physically we have to describe a series of acts which another person must perform in order to locate a point. Or in order to say what we mean by the distance between two points we must describe the operation of making the measurements necessary to determine the number we call the distance.

This description is in general very complicated. For example, in the case of distance it would involve nearly all the technique of practical astronomy. Moreover the

definition obtained is in general only approximate. For example, we never have an absolutely complete determination of what we mean by a point.

The physical definition of a point is notoriously without effect on the mathematical theory which follows. So far as the mathematical operations are concerned the point is purely and simply an undefined term. The abstract mathematical theory has an independent, if lonely, existence of its own. But when a sufficient number of its terms are given physical definitions, it becomes a part of a vital organism concerning itself at every instant with matters full of human significance. Every theorem can then be given the form "if you do so and so, then such and such will happen".

The places at which this life-blood of human meaning flows into the mathematical theory should, it seems to me, be the undefined terms. The process of making the physical application of the mathematical discipline will then consist simply in specifying the human operations by which we give meaning to the mathematically undefined terms. Each of the undefined terms will separate off a group of theorems in which the term appears, and these theorems will refer particularly to the set of physical operations involved in defining this term.

Thus in particular the question whether the plane as well as the line shall be an undefined term depends on how the plane is thought of physically. When solid objects and flat surfaces were fundamental physical objects, it was natural to think of the plane as a fundamental entity, somehow distinct from the set of points on it. But with particles and their paths playing the role which they do in modern physics it would seem more natural to regard lines and planes as sets of points.

I am not pretending to exhaust this question about the choice of undefined terms—only to indicate one respect in which the physical application seems to indicate a preference. In passing, however, I should like to refer to a remark

by the philosopher Geiger* who has made up postulates in terms of points, lines, planes and 3-spaces, of which last he postulates that there exists only one. Geiger's remark is that these are not the only "Elementargebilde", but that circles, parabolas, spheres, ellipsoids, and so on, also have a right to be considered. So "vom konsequenten Standpunkt aus" these should also appear as undefined terms. But for obvious reasons this program has not yet been carried out.

4. *Postulates for Analysis Situs.* The characterizations of an n-cell in terms of coordinates or of its correspondence with euclidean space have the disadvantage that they make use of ideas which are not invariant under the group of continuous transformations which we study in analysis situs. The undefined terms which we should expect to use are point and region, as R. L. Moore has already shown is possible for the two-dimensional case, or point and limit point.

To carry out the latter program successfully one should be able to start with the most general type of abstract set in which there is a definition of limit point, and then by successively excluding various types of sets, narrow down to the n-cell. This ought not to be very far out of the range of possibilities at the present time.

Another question about the foundations of analysis situs which deserves to be studied, and which can, I think, be studied with some hope of success, is the problem of a combinatorial set of axioms. This question has been before us with increasing clearness for some time. The researches of Poincaré made it evident that a large class of the fundamental problems of analysis situs are combinatorial in character. A combinatorial set of postulates was sketched in the Dehn-Heegaard article in the Encyclopädie. My Cambridge Colloquium lectures contain a treat-

* M. Geiger, *Systematische Axiomatik der Euklidischen Geometrie*, Augsburg, 1924.

ment in which combinatorial elements are separated out, but in which it is necessary to use the definition of an n-cell by means of correspondence with an ordinary geometric figure in order to obtain the proofs of invariance of the combinatorial constants. More recently Weyl has given a treatment from which he has excluded everything which he cannot obtain by combinatorial processes and into which the notions of "circuit" and of "partition" are brought by means of axioms instead of definitions.

There are several ways in which we can formulate the problem of a combinatorial analysis situs attack on the foundations of geometry. One of them would be this: Our undefined terms are 0-cells, 1-cells, 2-cells, and so on up to n-cells. We have relations of incidence between 0-cells and 1-cells, between 1-cells and 2-cells, and so on. We can in the three-dimensional case—but not yet in the four-dimensional one—state in abstract terms what conditions these incidence relations must satisfy in order that the complex shall be a manifold. Progress is being made —notably by Alexander—toward finding out what further conditions must be satisfied in order that this manifold shall be of a particular type, say a sphere or a 3-cell with its boundary. When we have this result we should be able to combine with the postulates so obtained further assumptions about the subdivision of the cells into complexes of cells and the combination of sets of cells to form single cells. From these we should be able to get the theorems of invariance and thus be able to prove all the theorems of analysis situs which can be formulated in terms of cells.

Such a treatment would put in evidence the properties of space which have to do with continuous deformations and indefinite subdivision, without carrying the subdivision to any limit. We should thus remain within the range of finite sequences of operations and avoid the difficulties which beset the theory of classes of large cardinal number. My impression is that Weyl has referred to this advantage

of the combinatorial attack on the foundations of geometry in a letter to me. The letter is unfortunately not at hand. In any case, it cannot be doubted that he has considered the question in connection with the doubts which he and Brouwer entertain as to the validity of the ordinary theory of the continuum.

The matter is also of interest from the physical point of view in that a theory of this sort could doubtless be worked out in terms of n-cells as the sole undefined objects. It would give a kind of n-dimensional continuum without necessarily implying the existence of any points. Some twenty years ago I had a similar project in mind, that of stating the foundations of geometry in terms of overlapping 3-cells or "chunks" of space, and discussed it a great deal with R. L. Moore without getting very far. Since then Huntington and Whitehead have given sets of postulates using solid portions of space as undefined elements. But they both have let the point in as a limiting case, whereas the true solution of the problem I have in mind would be a set of postulates which would lead to the main physical properties of space without providing a limiting process to define points. It would correspond to the true state of our physical knowledge of space, which is vague both for the infinitely large and also for the infinitely small.

5. *The Arithmetical Point of View.* While such an enterprise as that of which we have just been speaking is inspired by physical considerations, it would probably not be of any immediate interest to physicists. For the everyday uses of physics all the methods of approaching the Riemann geometry or the foundations of analysis situs must give way to the straight-forward statement that the points we mean to talk about are objects capable of representation by coordinates. Indeed, the definition of mathematics which will be most readily accepted by physicists is that which has been so cogently set forth

by Study* in various recent publications, that "mathematics includes computation with natural numbers and everything that can be founded upon it, but nothing else". This is a more inclusive account of mathematics than that given by Kronecker because Study allows the Dedekind cut process in building upward from the integers.

Whether we accept the arithmetical definition of mathematics or not, we have to admit that it gives us one way at least of treating the material in each of the branches of mathematics which we now regard as important. Moreover it puts us in touch at once with an indispensable process in each of these branches. Therefore from the point of view of teaching mathematics I think that the arithmetical method is bound to prevail. Most of the people who set out to learn mathematics do so with a view to some use which they are going to make of it. The arithmetical method will give them a maximum of mathematical information and power in a minimum of time. Everything that is irrelevant to this will have to be discarded.

I mean this statement to be as sweeping as possible. The treatment of elementary geometry modelled on that of Euclid is already being discarded. It will be replaced, not by a modern equivalent which stands apart as a logical structure whose chief methods are peculiar to itself, but by a properly developed analytic geometry. Likewise the synthetic treatment of projective geometry, however seductive it may be to some of us individually, will have to be left to one side as a study for specialists and as a theory which may be expounded to students at the stage when it is desirable for them to see what a well-rounded logical structure is like.

But the arithmetical method will be used for the primary exposition of all those theorems which we encounter in

* *Die Realistische Weltansicht und die Lehre vom Raum*, Braunschweig, 1914, and *Mathematik und Physik*, Braunschweig, 1923. Our quotation is from the latter book.

the main stream of mathematics. This will happen not because mathematics has been thus defined by any person however brilliant, but simply because life is too short for people who are not primarily mathematicians to learn mathematics otherwise.

I am tempted to add a remark that will not be so easily accepted by some of my friends who are devoted to analysis. Namely that much of the present theory of functions of one complex variable (which is really a conformal geometry incapable of generalization) will suffer the same fate as synthetic geometry. Its highly elegant but highly introversive methods will be left to one side for the specialist. The main current of mathematics will flow by, carrying with it only the more important facts of this theory; and these facts will be formulated as we are now learning to formulate them with the aid of analysis situs, in terms of the theory of functions of n real variables.

But while the arithmetical definition of mathematics specifies the actual content of present day mathematics and also corresponds to the most practicable way of expounding the subject to those who use it, it hardly seems adequate when we examine its foundations. If it is true that the basic ideas of geometry, mechanics, thermodynamics, etc., can all be defined in terms of real numbers, and real numbers in terms of natural numbers, it is also true that the natural numbers are mixed up with the basic ideas of formal logic in the most sticky fashion. Therefore the whole arithmetical structure is a portion of the edifice of formal logic which we are not as yet able to separate off. So, for example, the *Principia Mathematica* of Whitehead and Russell begins with the logic of propositions and goes far with the logic of classes before the natural numbers fully emerge. After that their treatment amounts essentially to the process of arithmetization.

But even after we are fully embarked on the arithmetical stage of the theory we are still beset by difficulties as to what we mean by a class of objects and what are the

allowable reasoning processes which can be applied to classes. We can neither obtain our definition of irrational number nor prove all the theorems desired, without reasoning about classes which are very different from those accessible to the simple experiments upon which the elementary logic is based. I mean experiments in counting and in identifying the same object at different times. The theory of infinite classes is obviously incomplete and beset with difficulties and paradoxes which have baffled the many excellent mathematicians who have attempted to set it in order. The latest and, I think, the most promising attempt to make it comprehensible is that of Hilbert who seems to propose that we apply our elementary logic to the symbols which are used in writing out the statements made in the logic required for a general theory of classes. It thus becomes a feasible problem to state what sorts of combinations of symbols give self-contradictory statements and then to study the possible combinations of symbols with a view to finding out by means of our elementary logic whether the postulates of the higher logic give rise to self-contradictory statements.

Whether or not this is a correct statement of Hilbert's program, the conclusion seems inescapable that formal logic has to be taken over by mathematicians. The fact is that there does not exist an adequate logic at the present time, and unless the mathematicians create one, no one else is likely to do so. In the process of constructing it we are likely to adopt the Russell view that mathematics is coextensive with formal logic. This, I suppose, is apt to happen whether we adopt the formalist or the intuitionalist point of view.

PRINCETON UNIVERSITY

RICCI'S COEFFICIENTS OF ROTATION*

BY HARRY LEVY

In a Riemann space whose linear element is given by the positive definite quadratic form

$$ds^2 = g_{ij}dx^i dx^j, \tag{1}$$

let us take n congruences of curves, defined by the equations

$$\frac{dx^1}{\lambda_h|^1} = \frac{dx^2}{\lambda_h|^2} = \cdots = \frac{dx^n}{\lambda_h|^n} \quad (h = 1, 2, \cdots, n). \tag{2}$$

Let us denote, as usual, the covariant components of congruence $\lambda_h|$ by $\lambda_{h|r}$, so that

$$\lambda_{h|r} = g_{rt}\lambda_h|^t. \tag{3}$$

We assume, further, that these n congruences are mutually orthogonal. We may then write

$$g_{rt}\lambda_h|^r\lambda_k|^t = \delta_{hk}, \tag{4}$$

where δ_{hk} is Kronecker's delta, that is,

$$\delta_{hk} = \begin{cases} 0, & h \neq k, \\ 1, & h \neq k. \end{cases} \tag{5}$$

Ricci† has found a set of invariants which he calls *coefficients of rotation* which are very important in the physical and geometric applications. They are defined by the equations

$$\gamma_{hij} = \lambda_{h|r,t}\lambda_i|^r\lambda_j|^t, \tag{6}$$

where $\lambda_{h|r,t}$ is the covariant derivative of $\lambda_{h|r}$,

$$\lambda_{h|r,t} = \frac{\partial\lambda_{h|r}}{\partial x^t} - \lambda_{h|p}\Gamma_{rt}^{p}, \tag{7}$$

* Presented to the Society, May 3, 1924.

† *Dei sistemi di congruenze ortogonali in una varietà qualunque*, MEMORIE DELLA R. ACCAD. DEI LINCEI, CLASSE DEI SCIENZE (5), vol. 2 (1896). See also *Méthodes de calcul différentiel absolu*, by Ricci and Levi-Civita, MATHEMATISCHE ANNALEN vol. 54 (1901), p. 147 ff.

where Γ_{rt}^{p} is the Christoffel symbol of the second kind of the fundamental form (1).

Lipka* has obtained the geometric interpretation of these functions γ_{hij} and the conditions that a congruence of curves be parallel in the sense of Levi-Civita.† In this paper I shall obtain Lipka's results in another way, and I shall also prove a new theorem.

Take an arbitrary curve, C, of congruence $\lambda_{j|}$ and fix on it some point P. Let $\mu_i|^r$ for $r = 1, 2, \cdots, n$ be the contravariant components of the vector which is parallel to itself along C and which at P coincides with the tangent vector to the curve of congruence $\lambda_{i|}$. Then, by the definition of parallelism, we have

$$\mu_i|^r{}_{,t}\lambda_j|^t = 0 \tag{8}$$

along C, where $\mu_i|^r{}_{,t}$ is the covariant derivative of $\mu_i|^r$. Let ω_{hi} be the angle, at the points of C, between the vectors $\lambda_{h|}$ and $\mu_{i|}$. Then

$$\cos\,\omega_{hi} = \lambda_{h|r}\mu_i|^r. \tag{9}$$

Differentiating covariantly with respect to x^t, we find

$$\frac{\partial \cos\omega_{hi}}{\partial x^t} = \lambda_{h|r,t}\mu_i|^r + \lambda_{h|r}\mu_i|^r{}_{,t}.$$

Multiplying by $\lambda_j|^t$, summing on t, and taking into account equations (8), we obtain

$$\frac{\partial \cos\omega_{hi}}{\partial s_j} = \lambda_{h|r,t}\mu_i|^r\lambda_j|^t, \tag{10}$$

where s_j is the arc of C. From (6) and from the definition of $\mu_{i|}$ it follows that at P we have

$$\left.\frac{\partial \cos\omega_{hi}}{\partial s_j}\right|_P = \gamma_{hij}.$$

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* *On Ricci's coefficients of rotation*, JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 3, 1924.

† *Nozione di parallelismo in una varieta qualunque*, RENDICONTI DI PALERMO, vol. 42 (1917). Also Bianchi's paper in RENDICONTI DI NAPOLI, (3a), vol. 27 (1922).

Hence γ_{hij} *is the rate of change along a curve of congruence* $\lambda_{j|}$ *of the cosine of the angle between the curves of* $\lambda_{h|}$ *and the parallels, with respect to* $\lambda_{j|}$, *to the curves of* $\lambda_{i|}$.

Since $\gamma_{hij} = -\gamma_{\omega hj}$,* the angle between the curves of $\lambda_{h|}$ and the parallels to $\lambda_{i|}$ changes at the same rate as the angle between the curves of $\lambda_{i|}$ and the parallels to $\lambda_{h|}$. In a euclidean space $\partial(\cos\omega_{hi})/\partial s_j$ is exactly what we would call the rotation of the congruence $\lambda_{i|}$ about $\lambda_{h|}$.

Suppose congruence $\lambda_{i|}$ forms a system of parallels with respect to congruence $\lambda_{j|}$, that is, those curves of $\lambda_{i|}$ which intersect one and the same curve of $\lambda_{j|}$ are parallel with respect to that curve, and this holds along all curves of $\lambda_{j|}$; then, $\mu_{i|}$ coincides with $\lambda_{i|}$, and

$$\cos\omega_{hi} = \delta_{hi}, \qquad (h = 1, 2, \cdots, n).$$

Hence

$$\gamma_{hij} = 0, \qquad (h = 1, 2, \cdots, n), \tag{11}$$

is the necessary condition that the congruence $\lambda_{i|}$ forms a system of parallels with respect to $\lambda_{j|}$. This is also sufficient. For from equations (6) we have

$$\lambda_{h|r,t} = \sum_{i,j=1}^{n} \gamma_{hij}\lambda_{i|r}\lambda_{j|t}.$$

By virtue of (3) and (4), (10) become

$$\frac{\partial\cos\omega_{ih}}{\partial s_j} = \sum_{l=1}^{n} \gamma_{lij}\lambda_{l|r}\mu_h{}^{|r} = -\sum_{l=1}^{n} \gamma_{lij}\lambda_{l|r}\mu_h{}^{|r}.$$

If (11) are satisfied, we have

$$\frac{\partial\cos\omega_{ih}}{\partial s_j} = 0, \qquad (h = 1, 2, \cdots, n).$$

Hence ω_{ii} is constant, and therefore $\lambda_{i|}$ is parallel along $\lambda_{j|}$.

The author has shown† that the necessary and sufficient conditions that a congruence $\lambda_{k|}$ have a family of m-dimensional hypersurfaces as orthogonal trajectories is that

* Ricci and Levi-Civita, loc. cit., p. 148.

† *Normal congruences of curves*, this BULLETIN, vol. 31, p. 39.

(12) $$\gamma_{hij} = \gamma_{jih}$$

where i takes on a definite set of $n-m$ values of the integers $1, 2, \cdots, n$ including the value k, and h and j take on those m values that i cannot assume. If each of $n-m$ congruences forms a system of parallels with respect to every one of the remaining m congruences equations (11) are satisfied for $h = 1, 2, 3, \cdots, n$; $i = i_1, i_2, \cdots, i_{n-m}$; $j = j_1, \cdots, j_m$; $i \neq j$, (12) is surely satisfied, and we have the following theorem.

THEOREM. *If each of $n-m$ congruences forms a system of parallels with respect to every one of the remaining m congruences, then the former have a family of m-dimensional hypersurfaces as orthogonal trajectories.* When $m = n-1$ this reduces to one of Lipka's theorems.

PRINCETON UNIVERSITY

SUR LES VALEURS ASYMPTOTIQUES DES COEFFICIENTS DE COTES

PAR J. OUSPENSKY

1. Parmi les formules de quadratures pour le calcul approché des intégrales définies la plus simple est, sans contredit, celle de Cotes, qui correspond à la division de l'intervalle d'intégration en parties égales. Supposons l'intervalle d'intégration $(0, 1)$ subdivisé en n parties égales; alors on peut déterminer $n+1$ constantes $A_0, A_1, A_2, \cdots, A_n$, nommées "coefficients de Cotes", de manière que la formule

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f\left(\frac{1}{n}\right) + A_2 f\left(\frac{2}{n}\right) + \cdots + A_n f(1)$$

soit exacte pour toute fonction $f(x)$ se réduisant à un polynôme d'un degré n'excédant pas $n-1$. Dans d'autres cas cette "formule de Cotes" n'est qu'approchée. Comme le degré d'approximation fourni par elle depend des valeurs numériques des coefficients $A_0, A_1, A_2, \cdots, A_n$, la

question sur les valeurs asymptotiques de ces coefficients pour n très grand n'est pas, ce me semble, dénuée de quelque interêt.

2. En posant

$$x_0 = 0, \quad x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad \cdots, \quad x_n = 1,$$

et

$$\varphi(x) = x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\cdots\left(x-\frac{n}{n}\right),$$

on a l'expression suivante pour le coefficient A_k, k étant un des nombres 0, 1, 2, $\cdots$, n:

$$A_k = \int_0^1 \frac{\varphi(x)dx}{(x-x_k)\varphi'(x_k)}.$$

Mais comme l'on sait que

$$\varphi'(x_k) = \frac{k}{n}\cdot\frac{k-1}{n}\cdots\frac{1}{n}\cdot\frac{-1}{n}\cdot\frac{-2}{n}\cdots\frac{-(n-k)}{n}$$
$$= (-1)^{n-k}\frac{\Gamma(k+1)\Gamma(n-k+1)}{n^n},$$

on trouve après quelques réductions faciles:

$$A_k = \frac{(-1)^{n-k}}{n\Gamma(k+1)\Gamma(n-k+1)}\int_0^n \frac{x(x-1)\cdots(x-n)}{x-k}dx.$$

Ainsi la question se réduit à trouver l'expression asymptotique pour n très grand de l'intégrale:

$$(1) \qquad J_n = \int_0^n \frac{x(x-1)\cdots(x-n)}{x-k}dx,$$

k étant un nombre quelconque de la suite 0, 1, 2, $\cdots$, n. En faisant usage des formules bien connues

$$x(x-1)\cdots(x-n) = (-1)^n\frac{\Gamma(x+1)\Gamma(n+1-x)\sin\pi x}{\pi},$$

$$\frac{\Gamma(x+1)\Gamma(n+1-x)}{\Gamma(n+2)} = \int_0^1 \xi^x(1-\xi)^{n-x}d\xi,$$

nous allons présenter d'abord J_n sous la forme d'une intégrale double:

$$J_n = \frac{(-1)^n \Gamma(n+2)}{\pi} \int_0^1 (1-\xi)^n d\xi \int_0^n e^{x \log \frac{\xi}{1-\xi}} \frac{\sin \pi x}{x-k} dx.$$

Ensuite, en utilisant la formule évidente

$$\int_0^n e^{x \log \frac{\xi}{1-\xi}} \frac{\sin \pi x}{x-k} dx = (-1)^k \left(\frac{\xi}{1-\xi}\right)^k \times \left[\int_0^{n-k} e^{x \log \frac{\xi}{1-\xi}} \frac{\sin \pi x}{x} dx + \int_0^k e^{x \log \frac{\xi}{1-\xi}} \frac{\sin \pi x}{x} dx\right],$$

et en transformant les intégrales du second membre à l'aide de la formule

$$\int_0^h e^{ax} \frac{\sin \pi x}{x} dx = \frac{\pi}{2} + \arctan \frac{a}{\pi} + \pi(-1)^{h-1} \int_{-\infty}^a \frac{e^{hx} dx}{\pi^2 + x^2},$$

que l'on établit aisément en supposant h un entier quelconque, on parvient à cette expression définitive de J_n:

$$J_n = (-1)^{n-k} \Gamma(k+1) \Gamma(n-k+1) + \Gamma(n+2) \times \left[(-1)^{n-1} \int_0^1 \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log \frac{1-\xi}{\xi}} \frac{e^{kx} dx}{\pi^2 + x^2} - \int_0^1 \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log \frac{1-\xi}{\xi}} \frac{e^{(n-k)x} dx}{\pi^2 + x^2}\right].$$

Nous allons faire usage de cette formule en supposant d'abord k différent de 0 et n. On voit de suite qu'en posant

$$(2) \quad \begin{cases} P_1 = \int_0^{\frac{1}{2}} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log \frac{1-\xi}{\xi}} \frac{e^{kx} dx}{\pi^2 + x^2}; \\ P_2 = \int_0^{\frac{1}{2}} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log \frac{1-\xi}{\xi}} \frac{e^{(n-k)x}}{\pi^2 + x^2} dx; \\ P_3 = \int_0^{\frac{1}{2}} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log \frac{\xi}{1-\xi}} \frac{e^{kx} dx}{\pi^2 + x^2}; \\ P_4 = \int_0^{\frac{1}{2}} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log \frac{\xi}{1-\xi}} \frac{e^{(n-k)x} dx}{\pi^2 + x^2} \end{cases}$$

on a

$$(3) \qquad J_n = (-1)^{n-k}\Gamma(k+1)\Gamma(n-k+1)+\Gamma(n+2) \times[(-1)^{n-1}P_1-P_2+(-1)^{n-1}P_3-P_4].$$

Mais comme

$$\int_{-\infty}^{\log\frac{\xi}{1-\xi}}\frac{e^{kx}\,dx}{\pi^2+x^2}<\int_{-\infty}^{\log\frac{\xi}{1-\xi}}e^{kx}\,dx=\frac{\xi^k}{k(1-\xi)^k},$$

$$\int_{-\infty}^{\log\frac{\xi}{1-\xi}}\frac{e^{(n-k)x}\,dx}{\pi^2+x^2}<\frac{\xi^{n-k}}{(n-k)(1-\xi)^{n-k}},$$

on voit immédiatement que

$$0<P_3<\frac{1}{k}\int_0^{\frac{1}{2}}\xi^n\,d\xi=\frac{1}{k(n+1)2^{n+1}};$$

$$0<P_4<\frac{1}{n-k}\int_0^{\frac{1}{2}}\xi^n\,d\xi=\frac{1}{(n-k)(n+1)2^{n+1}},$$

ce qui permet de présenter la formule (3) de cette manière:

$$(4) \qquad J_n=(-1)^{n-k}\Gamma(k+1)\Gamma(n+1-k)+\Gamma(n+2) \times\left[(-1)^{n-1}P_1-P_2+\frac{\Theta}{k(n-k)2^{n+1}}\right],\ -1<\Theta<1.$$

Toute la question se réduit ainsi à trouver les expressions asymptotiques de P_1 et P_2, ce qui présente un peu plus de difficulté.

3. La solution de cette question dépend en grande partie de la recherche de la valeur asymptotique de l'intégrale

$$K_n=\int_0^h\frac{(1-x)^n\,dx}{\left(\log\frac{1-x}{x}\right)^s}$$

pour n indéfiniment croissant, tandis que h est un nombre fixe, satisfaisant aux inégalités

$$0<h<\frac{1}{2},$$

et s un nombre entier positif donné. Considérons l'intégrale

un peu plus générale

$$K_n(\alpha) = \int_0^h \frac{\left(\frac{1-x}{x}\right)^\alpha (1-x)^n}{\left(\log\frac{1-x}{x}\right)^s} dx, \tag{5}$$

qui dépend d'un paramètre α et se réduit à K_n pour $\alpha = 0$. Nous allons supposer dans la suite que

$$-\frac{1}{2} \leqq \alpha \leqq 0.$$

L'inégalité presque évidente

$$0 < \int_0^h \frac{(1-x)^{n-\frac{1}{2}} x^{\frac{1}{2}} dx}{\left(\log\frac{1-x}{x}\right)^k} < \frac{1}{\left(\log\frac{1-h}{h}\right)^k} \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+2)},$$

et la formule asymptotique bien connue

$$\lim \frac{\Gamma(n+\beta)}{n^\beta \Gamma(n)} = 1, \text{ pour } n = \infty,$$

font voir que pour chaque valeur fixe de k, l'intégrale

$$\int_0^h \frac{(1-x)^{n-\frac{1}{2}} x^{\frac{1}{2}} dx}{\left(\log\frac{1-x}{x}\right)^k}$$

est de l'ordre $1/n^{3/2}$, c'est à dire qu'il existe une constante L telle que l'intégrale précédente soit $< L/n^{3/2}$.

Il suit immédiatement de cette remarque que les valeurs que prennent $K_n(\alpha)$ et ses dérivées pour $\alpha = -\frac{1}{2}$ se réduisent aux quantités d'ordre $O(1/n)^{3/2}$. Ensuite la dérivée d'ordre s de $K_n(\alpha)$ se present sous la forme

$$\begin{aligned} \frac{d^s K_n(\alpha)}{d\alpha^s} &= \int_0^h (1-x)^{n+\alpha} x^{-\alpha} dx \\ &= \frac{\Gamma(n+\alpha+1)\Gamma(1-\alpha)}{\Gamma(n+2)} - \int_h^1 (1-x)^{n+\alpha} x^{-\alpha} dx. \end{aligned} \tag{6}$$

L'intégrale qui figure dans le second membre de cette

formule est positive et plus petite que $(1-h)^{n-1/2}$; de l'autre côté les formules asymptotiques bien connues font voir que la quantité

$$\frac{\Gamma(n+\alpha+1)\,\Gamma(1-\alpha)}{\Gamma(n+2)}$$

peut être présentée ainsi

$$\frac{\Gamma(1-\alpha)}{n^{1-\alpha}}+\frac{\varphi(n,\alpha)}{n^{2-\alpha}},$$

$\varphi(n,\alpha)$ étant une fonction bornée de n et de α pour toutes les valeurs de n et pour toutes les valeurs de α comprises dans l'intervalle $(-\frac{1}{2}, 0)$. Il suit de ces remarques qu'on peut poser

$$\frac{d^s K_n(\alpha)}{d\alpha^s}=\frac{\Gamma(1-\alpha)}{n^{1-\alpha}}+\frac{\Psi(n,\alpha)}{n^{2-\alpha}},$$

où la fonction $\psi(n,\alpha)$ est de la même nature que la fonction $\varphi(n,\alpha)$ tout à l'heure mentionnée. On a d'après la formule de Taylor

$$K_n(\alpha) = K_n(-\tfrac{1}{2})+(\alpha+\tfrac{1}{2})K_n'(-\tfrac{1}{2})+\cdots$$
$$+(\alpha+\tfrac{1}{2})^{s-1}\frac{K_n^{(s-1)}(-\frac{1}{2})}{(s-1)!}+\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}K_n^{(s)}(x)\,dx.$$

Par ce qui précède on a uniformément en α

$$K_n(-\tfrac{1}{2})+(\alpha+\tfrac{1}{2})K_n'(-\tfrac{1}{2})+\cdots$$
$$+(\alpha+\tfrac{1}{2})^{s-1}\frac{K_n^{(s-1)}(-\frac{1}{2})}{(s-1)!}=O(1/n^{3/2}).$$

D'autre part en substituant au lieu de $K_n^{(s)}(\alpha)$ son expression (6) on obtient

$$\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\frac{d^sK_n(x)}{dx^s}\,dx=\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\frac{\Gamma(1-x)}{n^{1-x}}\,dx$$
$$+\int_{-\frac{1}{2}}^{\alpha}\frac{(\alpha-x)^{s-1}}{(s-1)!}\frac{\psi(n,x)}{n^{2-x}}\,dx.$$

La dernière intégrale est manifestement une quantité d'ordre $O(1/n^2)$ indépendamment de α. Quant à la première, l'appli-

cation répétée de l'intégration par parties conduit à ce résultat

$$\int_{-\frac{1}{2}}^{\alpha} \frac{(\alpha-x)^{s-1}}{(s-1)!} \frac{\Gamma(1-x)}{n^{1-x}}\,dx = \frac{\Gamma(1-\alpha)}{n^{1-\alpha}(\log n)^s}\left[1+\frac{\lambda(n,\alpha)}{\log n}\right]+O(1/n^{3/2}),$$

$\lambda(n, \alpha)$ désignant une fonction bornée de n et de α. Enfin nous arrivons à cette expression asymptotique de $K_n(\alpha)$:

$$K_n(\alpha) = \frac{\Gamma(1-\alpha)}{n^{1-\alpha}(\log n)^s}\left[1+\frac{A(n,\alpha)}{\log n}\right],$$

la fonction $A(n, \alpha)$ étant de même nature que $\lambda(n, \alpha)$. En posant $\alpha = 0$ nous voyons que

$$\int_0^h \frac{(1-x)^n\,dx}{\left(\log\dfrac{1-x}{x}\right)^s} \sim \frac{1}{n(\log n)^s},$$

ou d'une manière plus précise

$$(7) \qquad \int_0^h \frac{(1-x)^n\,dx}{\left(\log\dfrac{1-x}{x}\right)^s} = \frac{1}{n(\log n)^s}\left[1+\frac{\mu_n}{\log n}\right],$$

μ_n désignant une fonction bornée de n.

4. Considérons maintenant l'intégrale

$$\int_{-\infty}^{\omega} \frac{e^{hx}dx}{\pi^2+x^2},$$

où h est un nombre entier positif et ω un nombre positif quelconque. L'intégration par parties donne

$$(8) \qquad \int_{-\infty}^{\omega} \frac{e^{hx}dx}{\pi^2+x^2} = \frac{e^{h\omega}}{h(\pi^2+\omega^2)} + \frac{2}{h}\int_{-\infty}^{\omega} \frac{e^{hx}x\,dx}{(\pi^2+x^2)^2}.$$

En intégrant par parties de nouveau, nous avons

$$\int_{-\infty}^{\omega} \frac{e^{hx}x\,dx}{(\pi^2+x^2)^2} = \frac{e^{h\omega}\omega}{h(\pi^2+\omega^2)^2} + \frac{1}{h}\int_{-\infty}^{\omega} \frac{e^{hx}(3x^2-\pi^2)}{(\pi^2+x^2)^3}\,dx,$$

mais comme

$$\text{val. abs.} \int_{-\infty}^{\omega} \frac{e^{hx}(3x^2-\pi^2)}{(\pi^2+x^2)^3}\,dx < 3\int_{-\infty}^{\omega} \frac{e^{hx}dx}{(\pi^2+x^2)^2}$$
$$< \frac{3e^{h\omega}\omega}{(\pi^2+\omega^2)^2} + \frac{3}{h\pi^4} < \frac{3e^{h\omega}}{\omega^3} + \frac{3}{h\pi^4} < \frac{6e^{h\omega}}{\omega^3},$$

nous voyons que

$$\text{val. abs.} \int_{-\infty}^{\omega} \frac{e^{hx}x\,dx}{(\pi^2+x^2)^2} < \frac{7e^{h\omega}}{h\omega^3}.$$

Cette inégalité combinée avec la formule (8) nous conduit à cette formule utile

$$\int_{-\infty}^{\omega} \frac{e^{hx}dx}{\pi^2+x^2} = \frac{e^{h\omega}}{h(\pi^2+\omega^2)} + \frac{\mu e^{h\omega}}{h^2\omega^3}, \tag{9}$$

μ désignant une quantité dont la valeur absolue est toujours < 14.

5. Ces préliminaires établis, désignons par ξ_0 une racine de l'équation

$$\log\frac{1-\xi_0}{\xi_0} = 1,$$

qui est naturellement > 0 et $< 1/2$, et considérons l'intégrale

$$\int_0^{\xi_0} \xi^k(1-\xi)^{n-k}d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2}.$$

En utilisant la formule (9) nous aurons

$$\int_0^{\xi_0} \xi^k(1-\xi)^{n-k}d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2}$$
$$= \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2} + \frac{M'}{k^2}\int_0^{\xi_0} \frac{(1-\xi)^n}{\left(\log\frac{1-\xi}{\xi}\right)^3}d\xi.$$

Mais

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}$$
$$= \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\log\frac{1-\xi}{\xi}\right)^2} - \pi^2\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2\right)\left(\log\frac{1-\xi}{\xi}\right)^2},$$

et

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2\right)\left(\log\frac{1-\xi}{\xi}\right)^2} < \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\log\frac{1-\xi}{\xi}\right)^4},$$

donc, en tenant compte de la formule asymptotique (7), nous avons d'abord

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2} \sim \frac{1}{n(\log n)^2},$$

et ensuite nous voyons que le rapport de l'intégrale

$$\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\left(\log\frac{1-\xi}{\xi}\right)^3} \quad \text{à} \quad \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}$$

est asymptotique à

$$\frac{1}{\log n},$$

ce qui nous conduit à la formule

$$\int_0^{\xi_0} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx} dx}{\pi^2+x^2} = \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{N}{k\log n}\right],$$

où N pour des valeurs suffisamment grandes de n et quelque soit k reste au dessous d'une limite assignable, p. e., 14. D'un autre côté, on trouve immédiatement

$$\int_{\xi_0}^{\frac{1}{2}} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx} dx}{\pi^2+x^2} < \frac{1}{k}\int_{\xi_0}^{\frac{1}{2}} (1-\xi)^n d\xi < \frac{(1-\xi_0)^{n+1}}{k(n+1)},$$

et le rapport de cette quantité

$$\frac{(1-\xi_0)^{n+1}}{k(n+1)} \quad \text{à} \quad \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}$$

étant pour n suffisamment grand plus petit que, p. e.,

$$2(1-\xi_0)^{n+1}(\log n)^2,$$

peut être mis sous la forme

$$N'/(k\log n) \quad \text{où} \quad N' < 2n(1-\xi_0)^{n+1}(\log n)^2$$

tend vers zéro n grandissant indéfiniment. Ainsi nous avons enfin

$$\int_0^{\frac{1}{2}} \xi^k (1-\xi)^{n-k} d\xi \int_{-\infty}^{\log\frac{1-\xi}{\xi}} \frac{e^{kx}dx}{\pi^2+x^2} = \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{G}{k\log n}\right],$$

ou bien

$$P_1 = \frac{1}{k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{G}{k\log n}\right],$$

G désignant une quantité qui reste au dessous de 15 pour n suffisamment grand quelque soit k. De même

$$P_2 = \frac{1}{n-k}\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[1+\frac{G'}{(n-k)\log n}\right],$$

où G' est aussi <15 pour n assez grand quelque soit k. Il suffit maintenant d'introduire ces expressions dans la formule (4) pour avoir

$$J_n = (-1)^{n-k}\Gamma(k+1)\Gamma(n+1-k)+\Gamma(n+2) \times \left\{\int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2+\left(\log\frac{1-\xi}{\xi}\right)^2}\left[\frac{(-1)^{n-1}}{k}-\frac{1}{n-k}\right.\right.$$
$$\left.\left.+\frac{(-1)^{n-1}G}{k^2\log n}-\frac{G'}{(n-k)^2\log n}\right]+\frac{\Theta}{k(n-k)2^{n+1}}\right\}$$

et aussi

$$A_k = \frac{1}{n} + \frac{\Gamma(n+2)}{n\Gamma(k+1)\Gamma(n-k+1)} \times \left\{ \int_0^{\xi_0} \frac{(1-\xi)^n d\xi}{\pi^2 + \left(\log \frac{1-\xi}{\xi}\right)^2} \left[\frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k} + \frac{(-1)^{k-1} G}{k^2 \log n} + \frac{(-1)^{n-k-1} G'}{(n-k)^2 \log n} \right] + \frac{\Theta}{k(n-k)2^{n+1}} \right\}.$$

Le rapport de

$$\frac{\Theta}{k(n-k)2^{n+1}} \text{ à } \frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k}$$

est plus petit que

$$\frac{1}{2^{n+1}|n-k+(-1)^n k|} < \frac{1}{2^{n+1}},$$

car le nombre entier $n-k+(-1)^n k$ ne devient jamais égal à zéro. De même la valeur absolue du rapport de

$$\frac{(-1)^{k-1} G}{k^2 \log n} + \frac{(-1)^{n-k-1} G'}{(n-k)^2 \log n} \text{ à } \frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k}$$

pour n suffisamment grand, est plus petite que

$$\frac{15}{\log n} \cdot \frac{k^2+(n-k)^2}{k(n-k)|n-k+(-1)^n k|},$$

c'est à dire pour n pair

$$\frac{15}{\log n} \cdot \frac{k^2+(n-k)^2}{k(n-k)n},$$

et pour n impair

$$\frac{15}{\log n} \cdot \frac{k^2+(n-k)^2}{k(n-k)|n-2k|}.$$

Or quel que soit $k = 1, 2, 3, \cdots, n-1$,

$$\frac{k^2+(n-k)^2}{k(n-k)n} < 1, \ n \text{ pair}$$

et

$$\frac{k^2+(n-k)^2}{k(n-k)|n-2k|} < 3, \ n \text{ impair}$$

de sorte que le rapport précédent est toujours plus petit que

$$\frac{45}{\log n}$$

et tend vers zéro quand n grandit indéfiniment. De tout ce qui précède il est maintenant facile de déduire cette expression asymptotique de A_k:

$$A_k \sim \frac{\Gamma(n+1)}{n(\log n)^2 \Gamma(k+1)\Gamma(n-k+1)} \left[\frac{(-1)^{k-1}}{k} + \frac{(-1)^{n-k-1}}{n-k}\right],$$

valable en ce sens que le rapport des deux membres de cette égalité tend uniformément vers 1, quand n croît au delà de toute limite, k ayant une valeur quelconque $1, 2, 3, \cdots, n-1$. Quant aux coefficients A_0 et A_n, l'analyse toute semblable et même un peu plus simple donne

$$A_0 = A_n \sim \frac{1}{n \log n}.$$

Ainsi nous avons

$$A_0 \sim \frac{1}{n \log n}, \quad A_1 \sim \frac{1}{(\log n)^2}, \quad A_2 \sim \frac{-n}{4(\log n)^2} \text{ etc.}$$

et pour $n = 2l$, p. e.,

$$A_l \sim \frac{(-1)^{l-1} 4^l}{\sqrt{\pi}\, l^{5/2} (\log 2l)^2}.$$

On voit que les coefficients $A_2, A_3, \cdots, A_{n-2}$ tendent vers l'infini, ce qui fait voir que la formule de Cotes perd, toute valeur pratique tant que le nombre des ordonnées devient considérable.

TORONTO, 25 Août, 1924.

FUNCTIONS WITH AN ESSENTIAL SINGULARITY*

BY PHILIP FRANKLIN

1. *Introduction.* In this note we prove certain properties of functions possessing essential singularities. The results grew out of an attempt to prove that the equation

$$\sin x = x$$

has an infinite number of complex roots. This particular fact can be deduced very simply from Picard's theorem (see Theorem VI below), but it suggests other inferences which are much less immediate. The character of the theorems, which are given explicitly below, may be indicated by observing that they prove the existence of an infinite number of roots of the following typical equations:

$$x^3 e^{x^2} + 3x = 0,$$

$$\cos x - \frac{2x+1}{x^2-4} = 0,$$

$$\sec x + 5x = 0.$$

2. *A Corollary to Picard's Theorem.* One possible formulation of Picard's theorem is the following:†

If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is analytic except for a finite number of poles and has only a finite number of zeros, then the equation

$$f(x) - a = 0$$

has an infinite number of roots in the neighborhood in question for all values of a different from zero. The proposed corollary follows.

THEOREM I. *If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is*

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† Cf., e. g., Osgood, *Funktionentheorie*, 2d ed., vol. 1, p. 709.

analytic except for a finite number of poles and has only a finite number of zeros, while the function $A(x)$, not identically zero, is analytic except for poles in the complete neighborhood of P, then the equation

$$f(x)-A(x) = 0$$

has an infinite number of roots in this neighborhood.

To prove this, consider the function $f(x)/A(x)$. This function is analytic except for poles in the deleted neighborhood of P, and has an essential singularity at P. Further, it has at most a finite number of zeros and poles in the neighborhood in question, since both $f(x)$ and $A(x)$ have at most a finite number of zeros and poles there. Thus, by the theorem of Picard just stated, the equation

$$f(x)/A(x)-1 = 0$$

must have an infinite number of roots. Moreover, since $A(x)$ has only a finite number of poles, the equation

$$A(x)[f(x)/A(x)-1] \equiv f(x)-A(x) = 0$$

must in consequence have an infinite number of roots, which proves our contention.

3. *Applications to Entire Functions.* We may obtain a simplified statement of our theorem by specializing it somewhat. Let us take the point P at infinity, the deleted neighborhood as the proper plane, and require the function to have no poles. The function is then an entire function, and the theorem takes the following form.

THEOREM II. *If $E(x)$ is an entire function (not a polynomial) with only a finite number of zeros, and $R(x)$ is a rational function (not identically zero), the equation*

$$E(x)-R(x) = 0$$

always has an infinite number of roots.

If we further restrict $R(x)$ to be a polynomial, and notice that changing a finite number of the coefficients in the Maclaurin's expansion of a function is equivalent to adding

a suitably chosen polynomial to it, we obtain the following remarkable theorem.

THEOREM III. *If an entire function has at most a finite number of zeros, any function formed from it by changing a finite number of the coefficients in its Maclaurin's expansion has an infinite number of zeros.*

A statement equivalent to Theorem III is the following one.

THEOREM IIIa. *If an entire function is known to have only a finite number of zeros, and the coefficients of its Maclaurin's expansion from a certain point on are known, the preceding ones are uniquely determined.*

It is interesting to observe that the assumption that there actually exists an expansion with the desired properties is necessary for the truth of the theorem. That is, we can not start with any infinite expansion, and determine the earlier coefficients so as to give a function with only a finite number of zeros, as one might suppose. For example, if the terms following the mth are

$$K(x) = \sum_{m}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

no choice of the earlier terms will give a function which does not have an infinite number of zeros. This follows from Theorem V below, since $K(x)$ is the remainder in the expansion of $\sin x$.

4. *The Generalized Corollary.* There is an extension of our corollary, analogous to the extension of Picard's theorem to the case in which the function has an infinite number of poles and fails to take on two values more than a finite number of times. The extension is as follows.

THEOREM IV. *If the function $f(x)$ has an essential singularity at the point P, which is a cluster point of poles, otherwise being analytic in some neighborhood of P, and takes on two distinct values a and b at most a finite number of times in this neighborhood, while the function $A(x)$ is analytic*

except for poles in the complete neighborhood of P, and is not identically equal to a or b, then the equation

$$f(x)-A(x) = 0$$

has an infinite number of roots in the neighborhood in question.

To prove this, we first observe that the function

$$\frac{f(x)-a}{f(x)-b} = 1+\frac{b-a}{f(x)-b}$$

has an essential singularity at P (since a and b are distinct), and that it has at most a finite number of zeros and poles by our hypothesis as to $f(x)$. Likewise the function

$$\frac{A(x)-a}{A(x)-b}$$

is analytic except for poles in the complete neighborhood of P, and is not identically zero. Thus the equation

$$\frac{f(x)-a}{f(x)-b} - \frac{A(x)-a}{A(x)-b} = 0$$

satisfied the hypothesis of Theorem I, and hence has an infinite number of roots.

Next consider

$$\frac{[f(x)-b]\,[A(x)-b]}{a-b}\left[\frac{f(x)-a}{f(x)-b}-\frac{A(x)-a}{A(x)-b}\right] = 0.$$

Any value of x which satisfies the previous equation will satisfy this one, unless it is a pole of one of the outside factors. But such a value would have to be a pole of $f(x)$ or $A(x)$, and from the form of the previous equation, we see that it would have to be a pole of both, if of either. Thus all the roots lost are included in the poles of $A(x)$, and since this function has at most a finite number of poles, the equation last written has an infinite number of roots. Since this equation reduces to

$$f(x)-A(x) = 0,$$

the theorem is proved.

Both Theorems I and IV may be slightly strengthened by replacing the function $A(x)$ by two functions. The revised statement of Theorem IV is as follows:

THEOREM IVa. *If the function $f(x)$ has an essential singularity at the point P, which is a cluster point of poles, otherwise being analytic in some neighborhood of P, and takes on two distinct values a and b at most a finite number of times in this neighborhood, while the functions $A(x)$ and $B(x)$ are analytic except for poles in the complete neighborhood of P, and $A(x)/B(x)$ is not identically equal to a or b, then the equation*

$$B(x)f(x)-A(x) = 0$$

has an infinite number of roots.

This follows at once from Theorem IV, since that theorem may be applied to the equation $f(x)-A(x)/B(x) = 0$. The analogous strengthened form of Theorem I is somewhat similar to a theorem given by Borel for entire functions of finite genus, where our functions $A(x)$ and $B(x)$ are required to be polynomials.*

5. *Applications to Periodic Functions.* A periodic function which is not a constant must have an essential singularity at infinity. Hence the preceding theorems may be used to derive theorems about periodic functions. In fact, the periodicity enables us to weaken the hypothesis. We have, for example, the following theorem.

THEOREM V. *If $f(x)$ is periodic, is not a constant, and has no poles, while $A(x)$ is analytic except for poles in some complete neighborhood of infinity, and is not a constant, then the equation $f(x)-A(x) = 0$ has an infinite number of roots in the neighborhood in question.*

If the theorem were false, the equation

$$f(x)-A(x) = 0$$

would have only a finite number of roots, and its left member would satisfy all the conditions required of

* *Leçons sur les Fonctions Entières*, 1921, p. 90.

the function $f(x)$ of Theorem I. Likewise the function $A(x+p)-A(x)$, where p is the period of $f(x)$, would satisfy all the conditions required of the function $A(x)$ of that theorem. It is not identically zero since $A(x)$, being analytic or having a pole at infinity, and not being a constant, is not periodic. Thus by Theorem I, the equation

$$[f(x)-A(x)]-[A(x+p)-A(x)] = 0$$

would have an infinite number of roots. But, since

$$f(x) = f(x+p),$$

this equation is equivalent to

$$f(x+p)-A(x+p) = 0,$$

which accordingly has an infinite number of roots. Since this would necessitate the existence of an infinite number of roots for the equation

$$f(x)-A(x) = 0,$$

it contradicts our assumption, and the theorem is proved.

For one special choice of the function $A(x)$, we may weaken the restriction on $f(x)$ still further, so as to obtain the following theorem.

THEOREM VI. *If $f(x)$ is periodic and not a constant, the equation*

$$f(x)-ax = 0$$

always has an infinite number of roots.

For, if the equation $f(x)-ax = 0$ had only a finite number of roots, so would

$$f(x+np)-a(x+np) = 0,$$

p being the period of $f(x)$, and n being any integer. Since this is equivalent to $f(x)-ax = anp$, we would have a function with an essential singularity which failed to take on all the values anp more than a finite number of times, which contradicts Picard's theorem.

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THE CALCULUS OF VARIATIONS*

BY LEONIDA TONELLI

1. *The Purpose of this Note.* It is proposed to explain briefly here the method followed in my *Fondamenti di Calcolo delle Variazioni*†, of which the first two volumes have already been published. In these the simplest problem of the calculus of variations in the plane and the corresponding isoperimetric problem are considered, for integrands which are functions only of the coordinates and the inclination of the tangent at a point of the curve along which the integrals are taken; the other problems of the calculus of variations will be discussed in later volumes.

2. *Reasons for the Development of a new Method in the Calculus of Variations.* The inconveniences which present themselves in the classical methods of the calculus of variations are well known. Here I emphasize only the fact that, aside from the difficulties inherent in the verification for a given curve of the well known sufficient conditions for an extremum, there is further a serious difficulty which sometimes arrests at the very beginning the application of the classical results. It is that the theory of differential equations offers but few methods of deciding whether or not there exists an extremal joining two points which are not conveniently near to each other. I propose on that account to find a more direct procedure for the solution of the problems of the calculus of variations, independent of the theory of differential equations. At the same time my researches have two other objectives. The first is that of obtaining general results in the domain of absolute extrema, as contrasted with the classical method which is particularly devoted to relative extrema. In the second place it will be interesting to see if the new methods

* Translated into English and prepared for publication by G. A. Bliss.

† Bologna, Zanichelli; vol. 1, 1922; vol. 2, 1924.

are applicable to classes of curves which are more general than those which are usually considered in the treatises on the calculus of variations. One is necessarily restricted there to more special classes because the methods which are followed are founded essentially upon the consideration of extremals. The classical problem of Newton, the determination of a solid of revolution of least resistance, is a first example of the necessity of considering more general classes of curves than the ordinary ones.

3. *Fundamental Conceptions of the new Method.* Having thus specified the scope of the proposed investigation a natural method presents itself, that of the functional calculus according to the ideas of Volterra. The integrals which are considered in the calculus of variations, and which it is desired to make a maximum or minimum, depend for the most part upon curves or surfaces and are real valued functions of such curves or surfaces. These belong therefore in the domain of the functional calculus, and it is evident that the principal developments of the calculus of variations can be obtained by constructing, for its functions of curves and surfaces in the domain of the functional calculus, a theory of extrema corresponding to that which is usually developed for functions of one or more variables in the ordinary treatises on analysis. It is precisely this method which is followed here.

A first difficulty presents itself at once. In the ordinary theory of maxima and minima one starts from a fundamental property of the functions there considered, namely, *continuity*. But *continuity* cannot be made the basis of the new theory to be constructed, since the functions of curves and surfaces to be considered are in general discontinuous. For greater precision and in order to remain in the domain of the reasoning of the first two volumes of my *Fondamenti*, let us limit ourselves here to the functions of curves which are integrals whose integrands depend only upon the coordinates and the inclination of the tangent

at a variable point of the curve along which the integral is taken. If the property of continuity is required for such an integral, it is found that the only integrals which can be considered are those whose integrands are expressible linearly in terms of the slope of the curve of integration; and in the parametric case, the only integrals with this property are those which depend solely upon the coordinates of the end-points of the curve along which the integral is taken.

It is therefore necessary to discard *continuity*, and to place at the basis of the new theory a conception which is more general and better adapted to the functions which have to be considered.

The analysis of the concept of *continuity*, for the functions of numerical variables of the ordinary differential calculus, has already led Baire to decompose this concept into two others of a more elementary sort called *lower semi-continuity* and *upper semi-continuity*. The simultaneous possession of both these types of *semi-continuity* implies precisely ordinary *continuity*. Baire has also observed that the elementary reasoning which permits one to prove that a continuous function in a bounded closed domain has always a minimum or a maximum, is also sufficient to justify the same conclusion in the case when the function possesses lower or upper semi-continuity, respectively. This observation applied to our functions of curves, permits us to overcome the first serious difficulty and to establish the foundations of the new theory.

If the length of arc is taken as the variable of integration, and if the function to be integrated does not depend upon the derivatives of the coordinates of the points of the curve of integration, that is to say, if it is a function $\varphi(x, y)$ of the coordinates x and y, then an elementary geometric argument furnishes at once the result that when $\varphi(x, y)$ is always positive, its integral possesses lower semi-continuity as a function of the curve of integration. This property has already been noted in the memorable *Thesis* of Lebesgue. But is this semi-continuity a property only

of the integrals of the particular form here indicated, or does it belong to a class of integrals so large that it can be made the basis of a general method of the calculus of variations? My researches, begun in 1911, have permitted me to answer the latter part of this question affirmatively.

Among the integrals in the parametric form all those which Hilbert has called *regular* possess semi-continuity, as well as those which I have named *quasi-regular semi-normal,* and also those which are simply *quasi-regular* provided that another condition is added such as, for example, the condition that the function under the integral sign does not change sign. Besides these sufficient conditions for semi-continuity I have investigated also necessary conditions, and conditions which are at the same time necessary and sufficient. Conditions of this last sort for lower semi-continuity are that the integral be quasi-regular positive, and that every so-called *exceptional* point shall be the center of at least one circle such that the value of the integral along an arbitrarily chosen closed arc interior to the circle is always positive or zero. By an *exceptional* point is meant one at which the well known Weierstrassian function F_1 is identically zero in the derivatives x', y'.

For the case of a non-parametric integral a necessary and sufficient condition is simply that the integral be quasi-regular.

The semi-continuity hitherto considered is semi-continuity in the entire field, i. e., at every curve of the field. I have also studied semi-continuity at a particular curve and have arrived here at the conclusion that the property of semi-continuity corresponds to the well known condition of Weierstrass in the calculus of variations, a condition which, as is known, implies also that of Legendre. This conclusion seems to me interesting because it permits one to establish a complete correspondence between the conditions for a relative extremum in the differential calculus and those for the same sort of an extremum in the calculus of

variations. In the differential calculus, for a function $f(x)$ which is continuous and has derivatives of the first two orders, a point x_0 in the domain where $f(x)$ is defined furnishes a relative minimum for $f(x)$ if at x_0 the first derivative of the function is zero and the second derivative positive. In the calculus of variations a curve C_0, interior to the field in which the integral to be minimized is considered, furnishes a relative minimum for the integral if the first variation is zero and the second positive, and if furthermore the conditions of Weierstrass and Legendre are satisfied. But these conditions of Weierstrass and Legendre have precisely the effect of assuring us that the integral in question has lower semi-continuity at the curve C_0, a condition which in the case of the functions $f(x)$ is already implicitly contained in the hypothesis of continuity. Consequently to the conditions of the differential calculus relative to the vanishing of the first derivative and the positive sign of the second, correspond exactly in the calculus of variations, and with the same significance, the analogous conditions on the first and second variations. And the correspondence in the two fields between the conditions sufficient for a minimum is also repeated in the case of necessary conditions.

The study thus briefly summarized and the observation, which it may be well to repeat explicitly, that every integral must possess lower semi-continuity on each of its minimizing curves, have led me to construct a new development of the calculus of variations founded essentially on the concept of semi-continuity.

4. *Existence Theorems for Extrema.* A first application of semi-continuity can be made immediately to the demonstration of theorems concerning the existence of an absolute extremum, by following the same method of procedure as that of the differential calculus for extrema of ordinary functions of real variables.

Nevertheless a difficulty presents itself here which has

no analogue in the theory of extrema of the differential calculus. To establish the existence of an absolute minimum on a given interval for a function $f(x)$ which possesses lower semi-continuity, one uses, as is well known, a minimizing sequence $x_1, x_2, \cdots, x_n, \cdots$; such a sequence has at least one limit point by the theorem of Bolzano-Weirstrass; and in each one of these limit points the function $f(x)$, in virtue of its lower semi-continuity, assumes its minimum value. In the case of the calculus of variations the analogous minimizing sequence of curves $C_1, C_2, \cdots, C_n, \cdots$ does not always have a limiting curve because for classes of curves the theorem of Bolzano-Weierstrass is not always true. One has here the new circumstance that even though an integral possesses lower semi-continuity in a limited domain it may yet have no minimizing curve. To be able to reach a conclusion like that for the case of a function $f(x)$ it is necessary therefore to introduce some new condition which will assure the existence of a limiting element for the minimizing sequence. This further condition may assume various forms from which result numerous theorems of existence for extrema.

For integrals in parametric form I have begun, in the second volume of my *Fondamenti*, by demonstrating the existence of an extremum for the so-called quasi-regular definite integrals, a special case being the integral of a function $\varphi(x, y)$ which has always the same sign, a case which as I have already remarked was treated by Lebesgue in his *Thesis*. From the quasi-regular definite integrals I then pass to more general cases which require more profound analysis, seeking always to make the theory inclusive of the more important applications of the calculus of variations.

For the integrals in ordinary form the condition which assures the existence of a limiting element for a minimizing sequence takes a form different from that assumed in the parametric case, in that it is related to the behavior of the integrand function $f(x, y, y')$ as y' tends to infinity. Furthermore a variety of existence theorems are here stated

designed to satisfy as far as possible the exigencies of the applications, and results are found, as particular cases, which were established in other ways by Hadamard.

The theorems of existence, both for the parametric and ordinary integrals, are stated so as to secure the maximum generality possible with regard to the character of the curves themselves and the character of the class of curves which is to be considered. In the first place the curves are supposed only to be *rectifiable*; and in the second place the functions $y(x)$ representing them are supposed to be absolutely continuous according to the definition of Vitali, and such that along them the integral which it is wished to study exists and is finite. Furthermore, and this is indeed the most important point to set in evidence, the existence theorems are given almost all for the classes of curves which I have named *complete*. In the parametric case a *complete class* is simply a totality of rectifiable curves to which belong all those curves which are at the same time limiting curves of the class and rectifiable. For the ordinary non-parametric case one substitutes for rectifiability in this definition the two properties which I have indicated just above. The theorems of existence assume in this way the maximum of generality; they are completely independent of the use of extremal curves; and they have a form applicable also to problems of extrema which could not be approached in any way by already known methods.

It is important to notice that the method of demonstrating the existence of extrema by the use of semi-continuity permits further the consideration of classes of integrals which are not everywhere semi-continuous. Of this I have given examples in my book, studying those types of integrals which have been the subject of noteworthy researches by C. Carathéodory. Such integrals have in general extremizing curves with corner points. For an integral of this type I have given a demonstration of the existence of an extremum by reducing the problem to one with an integral which possesses everywhere the property of semi-continuity

and which is intimately related to the integral originally given. The same procedure permits me then to treat completely also the classical problem of Newton concerning the solid of revolution of minimum resistance, a problem whose solution, indicated by Newton himself, was completely justified only in 1900 by A. Kneser.

5. *Extremaloids and Extremals.* When the existence of an extremum has been proved it becomes necessary to pass to the discussion of the analytic properties of the extremizing curves. It is evident that these properties are intimately related to the character of the class of curves in which the extremum has been sought; and for the study of them the generality of the class considered must necessarily be restricted.

Limiting myself to the classes customarily considered in the calculus of variations, and moreover maintaining with respect to the character of the curves themselves all of the generality which I have indicated above, I am led to introduce the concept of an *extremaloid* which is a generalization of the known concept *extremal.*

In the parametric form, denoting by $F(x, y, x', y')$ the integrand function, I designate as an *extremaloid* every rectifiable curve satisfying the two equations

$$\int_0^s F_x ds - \frac{d}{ds}\int_0^s F_{x'} ds = c_1, \quad \int_0^s F_y ds - \frac{d}{ds}\int_0^s F_{y'} ds = c_2,$$

where s represents the length of arc. These equations are evidently generalizations of those of the extremals; they have the advantage of being in a form which permits them to be satisfied also by a curve which is only rectifiable. Generalizing a classical argument I prove that every extremizing curve, relative to all those of a given neighborhood of it joining the same two end points, is necessarily an *extremaloid* if it is interior to the field considered. I obtain thus the extension of Euler's condition to curves which are rectifiable only. I prove next that, for the *quasi-regular normal* integrals, the extremaloids are necessarily

extremals, and I have here the means of treating an important case in which the extremals present singularities arising from the vanishing of the coefficient of the term of the second order in their equation. For the case of the *quasi-regular normal* integrals again, I prove that for fields of a very general type, but with boundaries having no corner points pointing toward the interior of the field itself, all the extremizing curves have everywhere continously turning tangents even on the parts which touch the boundary. For special domains, or for unlimited domains when the behavior of the function F as the point (x, y) tends to infinity is suitably restricted, the extremizing curve turns out to be simply an extremal. *From this I derive theorems of existence for extremals joining two given points not necessarily near to each other, and for periodic extremals.*

I study further the uniqueness of the extremizing curves and of the extremal joining two given points.

It goes without saying that the known conditions for an extremum are re-deduced, those of Weierstrass and Legendre being, as has been already stated, an immediate consequence of the semi-continuity which must be verified on every extremizing curve.

I find moreover the known sufficient conditions for a relative extremum. Here I solve the problem first for a small region, reducing the study of the given integral to that of another integral possessing everywhere semi-continuity, and applying the results already obtained. The same method makes it possible to solve the analogous problem in the general case, which can also be treated by applying the results obtained for a small region.

For the non-parametric integrals greater difficulties in the case of Euler's condition present themselves, depending upon the fact that one can not be sure a priori of the existence of the integrals which occur in the equations of the extremaloids and whose integrands are the derivatives with respect to y and y' of the function $f(x, y, y')$ taken

at the points of the extremizing arc. The existence of these integrals seems probable, but I have not succeeded so far either in proving it for all cases or in constructing an example for which they do not exist. Nevertheless I have given in this connection propositions of a very general character. For example I have shown that every extremizing curve, relative to all those in a neighborhood of it joining two given points, is an extremaloid provided that it is interior to the field under consideration and has bounded difference quotients; and I have also shown that the condition relative to the difference quotients can be altogether suppressed for a noteworthy and extensive category of integrals which includes precisely those integrals which for the most part present themselves in the applications of the calculus of variations. For this same category of integrals and also for another still more general, I have moreover proved that every extremaloid is necessarily an extremal. I have studied especially *regular* integrals, and those which are *quasi-regular normal*, seeking to attain results analogous to those for the parametric case. Here the extremizing curves may have first derivatives of their ordinates which become infinite; but I have given conditions sufficient to insure that this shall not occur.

All the arguments made for the parametric case are taken up again for the integrals in non-parametric form; and by the method already indicated one finds here also the known necessary and the known sufficient conditions for a relative extremum. The problem which I have discussed hitherto is that of a non-isoperimetric extremum; but the method described is applied in the second volume of my *Fondamenti* to isoperimetric problems also.

The method of procedure, as is evident from what has been said, does not depend in any way upon the theory of differential equations; indeed it adds to that theory a contribution in the domain of the theorems of existence.

BOLOGNA, ITALY

SCHOUTEN ON RICCI-KALKÜL

Der Ricci-Kalkül. By J. A. Schouten. Berlin, Julius Springer, 1924. 312 pages.

This is Volume X of *Die Grundlehren der Mathematischen Wissenschaften,* edited by R. Courant, and is appropriately dedicated to G. Ricci, founder of the absolute differential calculus, in honor of his seventieth birthday. It is the second book on this subject published by Springer within two years. The authors (Struik and Schouten) are closely associated, and therefore the fact that the two books have many things in common is not surprising. The volume under discussion is more extended than Struik's, the additional space being devoted to a fuller discussion of non-Riemannian geometry. The methods are quite similar, but Schouten does not make use of quite so much symbolism, although Struik attributes most of his symbolism to Schouten.

These volumes give an exceedingly complete account of all the developments in differential geometry of the past five or six years, but the reviewer feels that much of the value, as reference books, is lost on account of the symbolism. The reading of many pages is often required for the full understanding of a single theorem, because it is impossible for one to keep these symbols in mind unless constantly using them.

Many of us feel that one of the best things Ricci did was to invent a consistent notation for covariant and contravariant systems. Ricci is always careful that the upper and lower indices shall have a significance. The ultra modern writers, following Weyl, have departed from this. The best example is $\Gamma^{\nu}_{\lambda\mu}$, which, according to the usual notation, should be a mixed system, covariant in λ, μ and contravariant in ν. Such is not the case, however. I, for one, fail to see why this confusion in notation should have been introduced, as it only makes the reading more laborious. One is compelled to investigate to see just what quantities are covariant or contravariant.

Geometers had been accustomed to think that Riemannian geometry was the most general geometry possible, but in the past few years has come the notion of parallel displacement, which leads to geometries quite different from that of Riemann. In the present volume, Schouten gives a very full account of these new geometries. In fact, only about fifty pages are devoted to Riemannian geometry. It does not take long to convince one that this book is entirely modern both as to material and point of view. Nearly every page has a reference to work which has appeared within six or seven years. A modern mathematical theory usually means one that is not more than fifty or

seventy-five years old, and this book, therefore, leaves the pleasant sensation of a really modern theory. The parts of the subject which are not modern are labeled so that one can easily recognise them.

The work is divided into seven chapters. The first gives, from a very general point of view, the algebraic part of the absolute calculus. The notions of a vector in a general space and the distinctions between covariant and contravariant vectors is carefully stated. In Chapter II, we begin the analytical part and are introduced at once to the notion of parallel displacement. The displacement is defined by the following properties:

I. The differential of a covariant, contravariant, or mixed quantity, which does not change with change of covariant scale, is a quantity of the same kind.

II. The differential is a linear function of the line elements or

$$\delta\Phi = \Phi_\mu dx^\mu;$$

Φ_μ is a quantity having one more covariant index than Φ and the operation which is represented is symbolized by ∇_μ, hence

$$\Phi_\mu = \nabla_\mu \Phi;$$

$\nabla_\mu \Phi$ is called the covariant derivative of Φ belonging to the displacement.

III. The differential of a sum is the sum of the differentials,

$$\delta(\Phi + \Psi) = \delta\Phi + \delta\Psi,$$

from which follows

$$\nabla_\mu(\Phi + \Psi) = \nabla_\mu \Phi + \Delta_\mu \Psi.$$

IV. The ordinary rules for the differential and therefore of the derivative of a product holds.

V. The differential of a numerical quantity (invariant) is the ordinary differential

$$\delta p = dp, \qquad \nabla_\mu p = \frac{\partial p}{\partial x^\mu}.$$

From these postulates the formulas for differentiation of vectors v^ν, w_λ are easily derived:

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\lambda\mu} v^\lambda,$$

$$\nabla_\mu w_\lambda = \frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^\nu_{\lambda\mu} w_\nu,$$

and similar formulas for quantities of higher order. The character of the displacement then depends on the quantities Γ, Γ', but instead of using them directly the following combinations are used.

Let

$$C_{\mu\lambda}^{\cdot\cdot\nu} = \Gamma_{\lambda\mu}^{\nu} + \Gamma_{\lambda\mu}^{\prime\nu},$$
$$S_{\lambda\mu}^{\cdot\cdot\nu} = \frac{1}{2}(\Gamma_{\lambda\mu}^{\nu} - \Gamma_{\mu\lambda}^{\nu}),$$
$$Q_{\mu}^{\cdot\lambda\nu} = \nabla_{\mu} g^{\lambda\nu}.$$

Then it is found that the displacement can be expressed in terms of C, S, Q, g, and the various kinds are classified according to the restrictions of C, S, Q. The affine displacement is characterized by

$$C = 0, \quad S' = \frac{1}{2}(\Gamma_{\lambda\mu}^{\prime\nu} - \Gamma_{\mu\lambda}^{\prime\nu}) = 0, \quad Q' = \nabla_{\mu} g_{\lambda\nu} = 0;$$

the Weyl displacement by

$$C = 0, \quad S' = 0, \quad Q = Q_{\mu} g_{\lambda\nu},$$

where Q_{μ} represents a vector; the Riemann displacement (defining the ordinary differential geometry) by

$$C = 0, \quad S' = 0, \quad Q' = 0.$$

For the general displacement a geodesic line is defined as a curve whose line elements are moved into line elements of the same curve and the differential equations of such a curve are

$$\frac{d^2 x^{\nu}}{dt^2} + \Gamma_{\lambda\nu}^{\nu} \frac{dx^{\lambda}}{dt} \frac{dx^{\mu}}{dt} = \alpha \frac{dx^{\nu}}{dt},$$

where α is a function of the coordinates. These have a marked similarity to the equations of geodesics in Riemannian geometry; and, in case $C = S' = Q' = 0$, they are identical with them.

Curvature is obtained by integrating δv^{ν} around an infinitesimal closed curve. The formula turns out to be very similar to that obtained by Levi-Civita for his parallel displacement. After fully developing these ideas Schouten then specializes the various kinds of displacements, devoting a chapter to each of the three mentioned above.

The whole structure is built up around the general idea of parallel displacement and cannot fail to convince the reader of the far-reaching importance of this notion. No geometer, unfamiliar with this subject, can afford to miss reading this book.

C. L. E. Moore

SHORTER NOTICES

Vorlesungen über Topologie. I. *Flächentopologie.* By B. von Kerékjártó. Volume 8 of the Courant Series. Berlin, Julius Springer, 1923. 8+270 pp. and 60 figures.

This production, from the pen of a young Hungarian mathematician, who is beginning to be known for his contributions to analysis situs, is welcome for several reasons. There are altogether too few books on the subject and one more is decidedly in place. It also gives under one cover a fairly complete treatment of the results obtained by Brouwer and his school on two dimensional topology, a useful thing indeed. The many and well chosen examples and figures are another good feature. For the sake of the average reader at least, we wish that the author had better amalgamated his material and introduced greater unity in his presentation. The *Topologie* will be especially useful to the reader familiar with point sets and wishing to learn more about their geometric applications, and also, say, in connection with Veblen's Colloquium Lectures.

The material in the book may be essentially classified into three groups: (a) Topology of the plane and its curves, centering around the Jordan curve theorems and including such questions as invariance of dimensionality and regionality, structure of regions and their boundaries, the general closed curve, etc. (b) Combinatorial analysis situs of two-dimensional manifolds. The treatment of this part is less felicitous than Veblen's in Chapter II of his Lectures. The author takes advantage of the relative simplicity of the theorems on plane Jordan curves, and thus more or less avoids questions of straightness. Unfortunately this method cannot be extended to three or more dimensions, and whereas Veblen's treatment of the two-dimensional case contains in germ, that of the more general one, Dr. Kerékjártó will be obliged to change front in his second volume, which is to be devoted to it. (c) Continuous transformations of surfaces (Brouwer's theorems on fixed points and the like), continuous families of curves on surfaces. This group of questions includes the author's more original and interesting contributions as well as the more recent results obtained by Brouwer and his students. If we mistake not, those on continuous families of curves are largely due to Dr. Kerékjártó himself.

The book concludes with a fair bibliography. The small number of American titles is doubtless to be ascribed to the lack of contact between the scientists of Central Europe and the rest of us—one of the most unpleasant things on the scientific landscape of the last few years.

S. Lefschetz

Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. By Adolph Kneser. 2d edition. Braunschweig, Vieweg, 1922. 8+292 pp.

Kneser's *Integralgleichungen*, which appeared first in 1911, was a pioneer in its field, and during the time which has elapsed since then, it has remained the only thing of its kind. To be sure, the book of Heywood and Fréchet emphasizes the physicial applications, and the problems of potential theory, in particular, have been methodically developed by means of integral equations. But the present work is, as far as the reviewer knows, unique in its inclusion of and emphasis on the application of the Hilbert-Schmidt theory to important one-dimensional problems.

The first edition was ably reviewed in the BULLETIN* by W. A. Hurwitz. The changes which have been made consist in about eighty pages of new or rewritten material, the remaining pages of the text having been reprinted without other change than an occasional alteration in notation. The most significant feature of the revision is the assembling and rounding out in one chapter (III) of a general theory of the symmetric kernel. The presentation follows the lines of the Schmidt theory, but it is further enhanced by the addition of paragraphs on Mercer's theorem on the convergence of the development of a continuous definite kernel in terms of its characteristic functions, and on Weyl's theorem on the diminishing effect on the positive characteristic numbers of a kernel caused by adding to the kernel a positive definite kernel.

The author finds a physical problem leading to an unsymmetric kernel in the vibrations of an elastic bar, account being taken of thermal effects. For this problem, the development question is treated by means of the theory of residues. Difficulties due to multiple characteristic numbers are not here encountered. There follow indications for a similar treatment of Sturm-Liouville problems in which the coefficients are not always necessarily real. The case in which the fundamental interval extends to infinity is then considered, and Hilb's generalization of the Fourier integral is derived and illustrated in the known cases of the Fourier integral and the corresponding integral in Bessel functions.

Another new topic arises from the treatment of a boundary value problem connected with electric cables, in which a boundary condition depends on the parameter. Here, ordinary orthogonality is replaced by a

* Vol. 19 (May, 1913), pp. 406–11. For other analyses and reviews, see *Lacour*, BULLETIN DES SCIENCES MATHÉMATIQUES, vol. 46 (1911), pp. 254–61; Korn, ARCHIV FÜR MATHEMATIK UND PHYSIK, (3), vol. 18 (1911), pp. 82–83; Plancherel, L'ENSEIGNEMENT MATHÉMATIQUE, vol. 13 (1911), pp. 428–29.

"weighted" (belastete) orthogonality: $V_1(1)V_2(1)+\int_0^1 V_1(x)V_2(x)dx=0$. Theorems analogous to the usual ones are indicated.

The bibliography in the appendix should not be overlooked. While it lays no claim to completeness, it is particularly valuable in the field of the applications. Its extent has been nearly doubled in the new edition.

The additions have materially enhanced the value of the book, and the chapter on the theory of the symmetric kernel has helped to meet Hurwitz' criticism as to the confusing effect of frequent alternation of general theory and particular examples. But only partially so, for as a rule several pages must be read before one can ascertain the precise conditions for the validity of a theorem, and in some cases the reader must bring an independent judgement to bear on his quest (e. g. on p. 38, line 19; the functions must be continuous and have *continuous* derivatives of first and second orders as well as piecewise continuous derivatives of third and fourth orders if the reasoning indicated is to establish the stated results). The style is further complicated by the habit of deferring the statement of a theorem until after its proof. Thus is imposed upon the attention of the reader the double task of following the reasoning and endeavoring to determine its import. Nor is much help given him by preliminary elucidation as to the general goal or the salient features of the discussion to follow.

The lacuna pointed out by Hurwitz in the proof of the theorem that to a solution of a homogeneous integral equation there always corresponds a solution of the associated equation has been allowed to stand; the section has been reprinted with such fidelity that a confusing typographical error recurs (p. 249, line 16: "Gleichung (7)" should read "Gleichung (3)").

The preparation requisite for a profitable reading of this book includes a knowledge of the rudiments of integral equations, some acquaintance with differential equations, with the theory of functions, and with physics; above all some mathematical maturity is essential. For one so equipped it is highly interesting and suggestive. Certainly no one who has to lecture on integral equations can afford to be unacquainted with its contents.

O. D. KELLOGG

Introduction à la Théorie de la Relativité, Calcul Différentiel Absolu, et Géométrie. By H. Galbrun. Paris, Gauthier-Villars et Cie., 1923. x+457 pp.

This work is a rather complete treatment of the mathematics of the relativity theory. Three chapters, about 100 pages, are devoted to a systematic and detailed exposition of the method of the absolute differential calculus. The differential geometry of n-dimensional space is allotted four chapters including slightly over 150 pages. The remainder of the book is devoted to mechanical and electromagnetic

theories, first from the older point of view, then from the standpoint of special relativity. Strangely enough the general theory of relativity is not specifically treated. So we miss the usual denouement of books on relativity—the three crucial tests.

In the very beginning of the exposition of the absolute calculus is an error which should be noted. The author, in his definition of invariant and co- and contravariant systems, has confused the term invariant as used by Ricci with the ordinary sense in which the word is used. With Ricci it is equivalent to the expression "a function which transforms by invariance", which is merely a statement of the transformations by which we are to transform the function and does not imply invariance under the transformations. In a remark following the definitions he seems to imply that for an ensemble of transformations forming a group this preservation of functional form, which his definitions demand in excess of Ricci's, is automatically taken care of and so need not be explicitly stated. As a matter of fact it would seem that in general functions satisfying his definitions do not exist. The upshot of the matter is that with this remark he throws overboard his definitions without once using them, and uses the standard definitions of Ricci throughout the book.

Several things unite to make this treatment perhaps the most valuable book of reference on the absolute calculus that has yet appeared. Very few things in development or proofs are left to the reader's imagination. The author even scorns to forget his signs of summation, as do most of the present generation. Covariant and contravariant derivatives are indicated by Δ's with subscripts and superscripts, and so are distinguishable from other tensors of the same order.

In the part of the book devoted to n-dimensional geometry we again find a valuable work for reference. The notion of a vector, with its co- and contravariant components, is basic in his treatment. Then the notion of parallel displacement of a vector, fundamental to the Weyl geometry, is twice explained in great detail, once for euclidian space, and later for the non-euclidian. Geodesics and curvature are defined in terms of parallel displacement as well as in the usual manner. Thus we are thoroughly prepared for the chapter on Weyl geometry which completes the geometrical part of the book.

Chapter IX is entitled Galilean spaces in rational mechanics and electromagnetic theory, and deals with the older theory but uses the tools of the new, and prepares for Chapter X on Special Relativity, and Chapter XI on the Memoir of Minkowski. Finally come some remarks on the kinematics of relativity, which help much towards clarity of thinking about the conceptions which serve to connect the mathematics and the physics of the relativity theory.

C. E. WILDER

Statique et Résistance des Matériaux. By Paul Montel. Paris, Gauthier-Villars, 1924. vi+273 pp.

The author states in the preface that the book represents a course given in the École des Beaux-Arts on statics and resistance of materials. The methods are almost wholly graphic and confined to problems in the plane. In the first one hundred and sixty-two pages, which are devoted to statics, the various methods for determining the stresses in a frame work are given and also graphical methods for the determination of the center of gravity and moments of inertia. In the chapters on resistance of materials, the main subjects considered are the theory of the beam, the buckling of a long thin rod under thrust, and arches.

The subject is approached in a neat way by calling attention to the difference between the theory of elasticity and the theory of the resistance of materials and the difference between the statically determinate and indeterminate problem is emphasized throughout. Before taking up frame works there is a chapter on reactions of the supports in which the different types of support are considered. The manner in which these things are done leaves the reader with the impression that the subjects treated have been presented more systematically than is generally the case in text books of this kind.

Two or three oversights may be noted. In connection with the discussion of the funicular polygon on page eighteen, the author speaks of two forces forming a couple but a couple is not defined till page thirty-nine. The stress diagram in Figure 64 should have the line df replaced by af. The statement at the bottom of page one hundred and seventy-nine that both theory and practice lead to the value $(2/5)E$ for the shearing modulus is open to question.

PETER FIELD

Binomial Factorizations, giving extensive Congruence-Tables and Factorization-Tables. By Lt.-Col. Allan Cunningham. London, Francis Hodgson. 1923. Vol. I. 96+287 pp.

Binomial Factorizations. By Lt.-Col. Allan Cunningham. London, Francis Hodgson. 1923. Vol. IV. 6+160 pp. Supplement to vol. I.

These two volumes are part of the outcome of thirty years of labor of the veteran computer. There are seven volumes of the work and the greater part is already printed off. The war has delayed the appearance of much of it. The present volumes contain an extensive list of the smallest root of congruences of the type $y^n+1 \equiv 0 \pmod{p^k}$ for various values of p, k and n. Thus there is a table, begun in the first volume and carried on into volume IV giving the smallest values of y satisfying the congruence $y^2+1 \equiv 0 \pmod{p}$ for available prime values

of p as high as the limit 99,989. Similar tables are also given for the smallest values of y satisfying the congruence

$$\frac{y^{2n}+1}{y^n+1} \equiv 0 \pmod{p}$$

for various values of n and various limits of p. Volume IV also contains tables of roots (x_1, x_2) of $x_1^n+x_2^n \equiv 0 \pmod{p}$ for various values of p and n.

The first volume contains also factorization tables giving the factors of a vast number of numbers of special forms. They will be found somewhat difficult to use on account of the new names that confront the reader on every page. One meets here not only "Pellians" with new and strange prefixes, but also "Aurifeullians" of various families. One stands a bit daunted before a "Dimorph-Bin-Aurifuellian". Out of this rather confusing mass of computation emerge, however, some usable tables giving values of y which make y^2+1, $(y^2+1)/2$, $(y^3-1)/(y-1)$, etc. take on prime values. There are also tables giving values of x, y which make x^n+y^n a prime or twice a prime. These tables go much beyond the limits of available factor-tables.

There is no room for doubt that Lt.-Col. Cunningham has undertaken and carried out an immense task, of value in the problem of identifying large primes, and in the breaking down of numbers of special forms into their prime factors.

D. N. LEHMER

The Calculus of Observations. A Treatise on Numerical Mathematics. By E. T. Whittaker and G. Robinson. London, Blackie and Son, Ltd., 1924. 16+395 pp.

"The present volume represents courses of lectures given at different times during the years 1913—1923, by Professor Whittaker to undergraduate and graduate students in the Mathematical Laboratory of the University of Edinburgh, and may be regarded as a manual of the teaching and practice of the Laboratory, complete save for the subject of Descriptive Geometry."

To the teacher of mathematics, in this country at least, a mathematical laboratory will be apt to suggest graphical and nomographical methods. In this book the work is almost entirely arithmetical, and we are told that in the University of Edinburgh graphical methods have almost all been abandoned "as their inferiority has become evident". This result of actual experience covering some ten years will perhaps come as a surprise to many teachers and practical computers to whom such methods appeal especially where great speed is very desirable and only rough approximations are necessary.

The first four chapters deal with the theory of interpolation, and are obtainable in a separate issue entitled *A Short Course in Inter-*

polation, published in 1923. They furnish a practical reference book on interpolation with equal and with unequal intervals, central difference formulas and applications. These four chapters are followed by a short one on Chio's method of computing the numerical value of a determinant.

The chapter on the numerical solution of algebraic and transcendental equations describes, illustrates and compares the more important methods. One might wonder if the value of an up to date calculating machine is fully appreciated at the Laboratory. The computation of the successive values satisfying a difference equation is especially well adapted for machine computation, and the successive values of S_p (the sum of the pth powers of the roots of an equation) can be ground out with uncanny readiness by one who has had a little experience. Then, using Bernoulli's method, the quotient S_{p+1}/S_p can be obtained to as many decimals as may be desired. The computing machine is also invaluable in constructing tables of differences.

There are also chapters on Numerical Integration, Normal Frequency Distributions, Least Squares, Fourier Analysis, Smoothing of Data, Correlation, Search for Periodicity and the Solution of Differential Equations. These subjects are all given with satisfying clearness and detail with plenty of examples, and sample solutions. The whole book presents evidence on every page of sound scholarship and good practical judgement. The authors call attention to the opportunities for research in the subject of numerical mathematics. "There is an evident need for new and improved methods of dealing with many of the problems discussed in the later chapters".

D. N. LEHMER

Les Lieux Géométriques en Mathématiques Spéciales avec Application du Principe de Correspondance et de la Théorie des Charactéristiques à 1,400 Problèmes de Lieux et d'Enveloppes. By T. Lemoyne. Paris, Vuibert, 1923. 146 pp.

This little pamphlet summarizes Chasles' theory of characteristics together with the extensions which the author (with Brocard) developed in volume I of *Courbes Géométriques Remarquables*, and gives 1400 problems by way of illustration. It will be recalled that the characteristics μ, ν of a system of conics refer to the number of conics which pass through an arbitrary point, and are tangent to an arbitrary line, respectively. The characteristics of 170 systems of conics, and of 41 systems of circles are listed. Fundamental theorems give formulas for the order or class of many loci or envelopes connected with these systems in the form $\alpha\mu + \beta\nu$. Chasles gave 32 such fundamental theorems for the values of α and β; Lemoyne adds about 50 more, either with proofs or with references to the *Courbes Géométriques*

Remarquables, an important addition being to certain specializations where Chasles' formulas are inapplicable. The author is then in a position to state, not to be sure 80 times 170 results, for not every fundamental theorem and listed entry of characteristics may be paired, but perhaps 10,000 results (problems). Of these he actually gives about 1400. For two or three theorems he feels compelled to go practically through the 170 entries. Otherwise it is fair to say that he has chosen pretty much at random among those cases where the order or class is low. In some cases the order or class may be extremely large. Thus the locus of the vertices of conics tangent to three given conics and to a given line is a curve of order 1000. But the author does not gloat over these millics.

Lack of system seriously impairs the value of this book. The table of characteristics is extremely awkward for reference; failure to number the entries seems inexcusable. If the reader will number these he will find slips in 100, 102, 133, 134, 135, and 159. Among the problems there are obvious slips in 47, 148–152, 206, 248, 250, 690, 900, 1168, and 1291, while 605, 606, 607 are repetitions of 597, 598, 599, and 1141—except for a non-significant variation in punctuation—of 1121. There are unnecessary and irritating changes in notation, such as the use, in problem 1288, of D to denote both an asymptotic direction and an arbitrary line, while other capital letters stand for points. On the other hand, the enumerative results in these problems have all been checked and have been found correct.

B. H. Brown

Aufgabensammlung zur Funktionentheorie. By Konrad Knopp. Berlin and Leipzig, Walter de Gruyter. 1923. 135 pp. I. Teil: *Aufgaben zur elementaren Funktionentheorie.* Sammlung Göschen.

This little book contains an excellent collection of problems on the elementary theory of complex functions. They are listed by chapters under six headings, fundamental concepts, sequences of numbers and infinite series, functions of a complex variable, integral theorems, developments in series, conformal mapping. Each of these chapters is further divided into two or three sections.

It has been said that if we could have excellent sets of problems published separately from texts, the task of text-book writers would be greatly simplified and they could concentrate on the presentation of their subject alone. The problems in this book could be a companion to any text which might be written on the subject of the theory of functions of a complex variable, covering the field of a single variable.

H. J. Ettlinger

NOTES

The third number of volume 26 of the TRANSACTIONS OF THIS SOCIETY (July, 1924) contains the following papers: *The general theory of a class of linear partial q-difference equations,* by C. R. Adams; *The summability of the triple Fourier series at points of discontinuity of the function developed,* by B. M. Eversull; *An unusual type of expansion problem,* by M. H. Stone; *On the independence of principal minors of determinants,* by E. B. Stouffer; *A necessary and sufficient condition that two surfaces be applicable,* by W. C. Graustein and B. O. Koopman; *Extensions of relative tensors,* by O. Veblen and T. Y. Thomas; *Geometries of paths for which the equations of the paths admit a quadratic first integral,* by L. P. Eisenhart; *A general mean-value theorem,* by D. V. Widder.

The concluding number of volume 26 of the TRANSACTIONS OF THIS SOCIETY (October, 1924) contains: *Algebras which do not possess a finite basis,* by J. H. M. Wedderburn; *Determination of all the prime power groups containing only one invariant subgroup of every index which exceeds this prime number,* by H. A. Bender; *Invariants of the linear group modulo* $\pi = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_n^{\lambda_n}$, by C. Gouwens; *A uniqueness theorem for Legendre and Hermite polynomials,* by K. P. Williams; *A new type of class number relations,* by E. T. Bell; *A new method in the equivalence of pairs of bilinear forms,* by R. G. D. Richardson; *Relative extrema of pairs of quadratic and hermitian forms,* by R. G. D. Richardson.

The concluding number of volume 46 of the AMERICAN JOURNAL OF MATHEMATICS (October, 1924) contains: *A general class of problems in approximation,* by D. Jackson; *On a rational plane quintic curve with four real cusps,* by P. Field; *Projective properties of a ruled surface in the neighborhood of a ruling,* by A. F. Carpenter; *Complete characterization of dynamical trajectories in n-space,* by L. M. Kells; *Rods of constant or variable cross section,* by C. A. Garabedian.

The first of the CARUS MATHEMATICAL MONOGRAPHS (see this BULLETIN, vol. 28, p. 74) has been published recently by the Open Court Publishing Company under the auspices of the Mathematical Association of America. The general purpose of this series is to contribute to the dissemination of mathematical knowledge. The present volume, a book of 189 pages, is by Professor G. A. Bliss on the topic *Calculus of Variations*; it will be reviewed promptly in this BULLETIN. Members of the Association may secure it at a reduced price through the

Secretary, Professor W. D. Cairns. It is announced that a second monograph, by Professor D. R. Curtiss on *Functions of a Complex Variable*, will appear in the near future.

The University of London has awarded its Sir John William Lubbock memorial prize in mathematics to Miss E. M. Jones, of Royal Holloway College.

On October 5, 1924, a tablet was unveiled at the birthplace of Charles Hermite, at Dieuze, France. Professors Henri Andoyer and Émile Picard represented the Paris Academy of Sciences at the ceremony.

Professor R. Weitzenböck, of the University of Amsterdam, has been elected a member of the Amsterdam Academy of Sciences.

Professor A. V. Vassilief, formerly of the Universities of Kazan and Petrograd, celebrated on December 12, 1924, the fiftieth anniversary of his entrance into teaching. Professor Vassilief was president of the Physico-Mathematical Society of Kazan at the time of the celebration of the centenary of Lobachevsky and the foundation of the Lobachevsky prize.

On the occasion of the meeting of the British Association for the Advancement of Science and the Mathematical Congress, the University of Toronto conferred honorary degrees on Sir David Bruce, Professor G. Koenigs, Sir Charles Parsons, Sir John Russell, Sir Ernest Rutherford (as announced in an earlier issue of this Bulletin), and Professors F. Severi, V. Stekloff, and C. de la Vallée Poussin.

The one hundreth anniversary of the announcement by Sadi Carnot of the second law of thermodynamics was celebrated in New York City under the auspices of the Engineering Foundation, on December 4, 1924. Addresses were delivered by Professor M. I. Pupin, of Columbia University, and Dr. W. L. Emmet, of the General Electric Company, on *Carnot's principle*.

On the occasion of the centenary of the Franklin Institute, the University of Pennsylvania conferred honorary degrees on Sir W. H. Bragg, Mr. W. C. L. Eglin, Dr. Charles Fabry, Sir Charles Parsons, Dr. E. W. Rice, Jr., and Dr. Pieter Zeeman.

The General Electric Company has appropriated to Union College a fund of $25,000 in memory of Dr. C. P. Steinmetz. The income is to be used for scholarships.

The Rensselaer Polytechnic Institute, of Troy, the oldest school of science and engineering in any English-speaking country, celebrated on October 3-4, 1924, the centenary of its foundation. Dr. G. A. Campbell, of the American Telephone and Telegraph Company, represented the American Mathematical Society at the ceremonies. On this occasion

the Institute conferred the honorary degree of doctor of science on Professor A. A. Michelson, president of the National Academy of Sciences.

Professor A. A. Michelson was appointed one of the official representatives of the United States at the third Pan-American Congress, held at Lima, Peru, December 20, 1924—January 6, 1925. The delegates of the American Mathematical Society to this congress were Professors E. V. Huntington and D. N. Lehmer (see this BULLETIN, vol. 30, p. 566), but Professor Lehmer was unable to attend.

Professor A. Einstein will deliver a series of lectures on relativity at Buenos Aires in the spring of 1925, at the invitation of various Argentine scientific bodies.

Professor A. N. Whitehead, of Harvard University, will deliver a series of lectures on *Science and the modern world* under the auspices of the Lowell Institute, of Boston, beginning February 2, 1925.

Professor E. G. Bill, of Dartmouth College, has been appointed chairman of a committee created by the College Entrance Examination Board to consider the desirability of psychological tests to be given by that board.

Dr. Gustav Doetsch, of the University of Halle, has been called to a professorship at the Stuttgart Technical School.

Professor H. A. Wilson, of Rice Institute, has been appointed professor of natural philosophy at Glasgow University.

At St. Olaf College, Northfield, Minnesota, Professor Martin Nordgaard, formerly head of the department of mathematics at Antioch College, has been appointed professor and head of the department of mathematics, and Mr. Arthur Solum of the University of Minnesota has been appointed assistant professor.

At Union University, Schenectady, Professor S. A. Rowland and Mr. L. A. DeRonde have been granted leave of absence and are studying at Chicago and Harvard, respectively.

Mr. G. F. Abrich has been appointed assistant professor of mathematics at Des Moines University.

Dr. G. N. Bauer, has been appointed associate professor of mathematics at the University of New Hampshire.

Professor F. P. Brackett of Pomona College has been granted leave of absence for the first semester of 1924–1925, and is travelling in Europe.

Assistant Professor F. W. Bubb, of Washington University, St. Louis, is on leave of absence for the academic year 1924–1925 and is studying at the University of Chicago as a Research Council fellow in physics.

Miss Mary Campbell, of the University of Texas, has been appointed professor and head of the department of mathematics at South Park Junior College.

Assistant Professor L. J. Comrie of Swarthmore College has accepted a position in the observatory at Northwestern University.

Mr. Victor Doshkess, of Lafayette College, has been promoted to an assistant professorship of mathematics.

Mr. H. K. Fulmer, of the Georgia School of Technology, has been promoted to an assistant professorship of mathematics.

Dr. G. W. Hess has been appointed professor of mathematics and head of the department at Union University, Jackson, Tennessee.

At Lehigh University, Professor P. A. Lambert has been made head of the department of mathematics and astronomy.

Mr. W. H. Lyon has been appointed assistant professor of mathematics at the Kansas State Agricultural College.

Professor H. B. Mitchell, of Columbia University, has resigned to enter business.

Mrs. R. B. Montgomery, of Lynchburg College, has been promoted to an assistant professorship of mathematics.

Miss Inez Morris has been appointed assistant professor of mathematics at the Indiana State Normal School, Terre Haute.

Dr. J. O. Osborn, of Cornell University, has been appointed teacher of mathematics at Harris Teachers College, St. Louis.

Dr. W. P. Ott, of Vanderbilt University, has been appointed head of the department of mathematics at the University of Alabama.

Dr. G. T. Pugh has been appointed professor of mathematics at Winthrop College.

At Catholic University, Associate Professor O. J. Ramler has been promoted to a full professorship of mathematics.

Mr. P. L. Rea, of Marietta College, has been promoted to an assistant professorship of mathematics.

Professor J. E. Rowe, of the College of William and Mary, has been made Director of Extension.

Mr. C. A. Rupp, of Hamline University, has been appointed adjunct professor of mathematics at the University of Texas.

Mr. J. H. Synnerdahl, of Lake Forest College, has been appointed assistant professor of mathematics at Park College, Parkville, Missouri.

Dr. J. S. Taylor, of the Massachusetts Institute of Technology, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Miss Velma Tisdale, of Southwestern University, Georgetown, Texas, has been promoted to an assistant professorship of mathematics.

Mr. W. M. Whyburn, of South Park Junior College, has been

appointed assistant professor of mathematics at the Agricultural and Mechanical College of Texas.

Dr. R. L. Wilder, of the University of Texas, has been appointed assistant professor of mathematics at Ohio State University.

Mr. W. L. Williams has been appointed adjunct professor at the University of South Carolina.

Professor Helge von Koch, of the University of Stockholm, died March 11, 1924.

The death is announced of Professor August Föppl, of the Munich Technical School, in his seventy-first year.

Professor Karl Wieghardt, of the Dresden Technical School, died June 11, 1924, at the age of fifty.

Mr. F. H. Bradley, fellow of Merton College, Oxford, and writer on logic and philosophy, died September 18, 1924, at the age of seventy-eight.

Dr. J. E. Campbell, F. R. S., fellow and tutor of Hertford College, Oxford, and mathematical lecturer at University College, Oxford, died October 1, 1924.

Professor C. B. Austin, of Ohio Wesleyan University, died September 9, 1924, at the age of seventy-three.

Professor E. L. Larkin, director of Mount Lowe Observatory, died October 11, 1924, at the age of seventy-seven.

The following have been appointed instructors in mathematics:
Athens College (Georgia), Miss Nancy L. Moorefield;
Brown University, Mr. F. C. Jonah;
University of Buffalo, Mr. H. D. Highet;
Case School of Applied Science, Mr. J. E. Merrill;
Georgia School of Technology, Messrs. G. S. Bruton, G. W. Nicholson, L. K. Patton, H. H. Pixsley, D. P. Richardson;
Hamline University, Mr. E. D. Wells;
University of Kansas, Mr. R. H. Marquis;
Lehigh University, Mr. C. A. Balof;
University of Maine, Mr. E. H. Hadlock;
Mercer University, Messrs. C. H. Berryman and J. L. Talley;
University of Michigan, Messrs. W. C. Green, A. P. Maslow; J. A. Shohat, G. S. Van Fleet;
Purdue University, Mr. J. C. Bennett;
University of South Carolina, Mr. R. L. Jones;
Southwestern University, Georgetown, Texas, Mr. J. C. Chaney;
Swarthmore College, Mr. D. B. McLaughlin;
University of Texas, Miss Elizabeth T. Stafford;
Union University (Schenectady), Messrs. L. W. Hussey and E. E. Steinert;
Washington State College, Miss Eleanor E. Boyd;
Washington University (St. Louis), Messrs. H. R. Grumann and E. H. Lund;
College of William and Mary, Miss Elizabeth Mercer;
University of Wisconsin, Mr. H. S. Pollard;
Wofford College, Mr. W. C. Herbert.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ANDREWS (E. S.) and HEYWOOD (H. B.). The calculus for beginners. 2d revised edition. London, Scott, Greenwood and Son, 1924. 284 pp.

APPELL (P.). See ZORETTI (L.).

AUBRY (L.). Démonstration du postulat d'Euclide sur les parallèles. Paris, Gauthier-Villars, 1924. 12 pp.

BALDUS (R.). Formalismus und Intuitionismus in der Mathematik. Karlsruhe i. B., G. Braun, 1924. 45 pp.

BERTINI (E.). Einführung in die projektive Geometrie mehrdimensionaler Räume mit einem Anhang über algebraische Kurven und ihre Singularitäten. Nach der zweiten italienischen Auflage deutsch herausgegeben von A. Duschek. Wien, Seidel, 1924. 22+480 pp.

BUSCO (P.). Les cosmogonies modernes et la théorie de la connaissance. Paris, Alcan, 1924. 436 pp.

CAHEN (E.). Théorie des nombres. Tome II: Le second degré binaire. Paris, Hermann, 1924. 10+736 pp.

CHATELET (A.) et KAMPÉ de FÉRIET (J.). Calcul vectoriel. Théorie. Applications géométriques et cinématiques. Paris, Gauthier-Villars, 1924. 426 pp.

DINGLER (H.). Die Grundlagen der Machschen Philosophie mit Erstveröffentlichungen aus seinen wissenschaftlichen Tagebüchern. Leipzig, Barth, 1924. 106 pp.

DUSCHEK (A.). See BERTINI (E.).

FOUET (E. A.). Leçons de géométrie élémentaire. Paris, Vuibert, 1924. 16+348 pp.

HARDING (A. M.) and MULLINS (G. W.). Analytic geometry. New York, Macmillian, 1924. 8+312 pp.

HARDY (G. H.). Orders of infinity. The "Infinitärcalcul" of Paul du Bois-Reymond. (Cambridge Tracts, No. 12.) Cambridge, University Press, 1924. 6+77 pp.

HEYWOOD (H. B.). See ANDREWS (E. S.).

KAMPÉ DE FÉRIET (J.). See CHATELET (A.).

KING (L. V.). On the direct numerical calculation of elliptic functions and integrals. Cambridge, University Press, 1924. 8+42 pp.

KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. 3te Auflage. Band 1. (Die Grundlehren der mathematischen Wissenschaften, Band 14.) Berlin, Springer, 1924. 320 pp.

KNOPP (K.). Theorie und Anwendung der unendlichen Reihen. 2te erweiterte Auflage. (Die Grundlehren der mathematischen Wissenschaften, Band 2.) Berlin, Springer, 1924. 10+526 pp.

LICHTENSTEIN (L.). Neuere Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. (Sonderabdruck aus der Encyklopädie der mathematischen Wissenschaften.) Leipzig, Teubner, 1924. 58 pp.

LIETZMANN (W.). Methodik des mathematischen Unterrichts. Teil 3: Didaktik der angewandten Mathematik. Leipzig, Quelle und Meyer, 1924. 12+234 pp.

MEHMKE (R.). Leitfaden zum graphischen Rechnen. 2te, vermehrte und verbesserte Auflage. Leipzig und Wien, Deuticke, 1924. 8+185 pp.

MOREUX (T.). Pour comprendre le calcul différentiel. Paris, Doin, 1924. 238 pp.

—— Pour comprendre la géométrie dans l'espace. Paris, Doin, 1923. 7+209 pp.

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